Chicken Nuggets

Question

At a fast-food take-away you can order chicken nuggets in boxes containing six, nine or twenty nuggets. What is the largest number of nuggets you *cannot* buy with these boxes?

Solution

Clearly, all multiples of 6 can be bought, so we only need to consider numbers whose remainders are 1 to 5 after division by 6. This suggests the use of arithmetic modulo 6.

Any integer can be expressed as 6n+r, where n is an integer and r is 0, 1, 2, 3, 4 or 5. If the number of boxes we can buy is expressed as 6u+9v+20w, where u, v and w are integers, then we can say that 6u+9v+20w = 6n+r. Expressing this in arithmetic modulo 6, we have $(9v+20w)_{modulo6} = r$.

For some numbers with a given remainder modulo 6 there are no combinations of boxes that we can buy. For example, we cannot buy exactly 8 nuggets. 8 has remainder 2 when divided by 6. However, we *can* buy 26 nuggets. 26 also has remainder 2 when divided by 6. We can buy all numbers larger than 26 that have remainder 2 when divided by 6 simply by buying extra boxes of 6. So somewhere between 8 and 26 is the *smallest* number with remainder 2 than we can buy (20, in fact). The number 6 less than this smallest number must be the largest number with remainder 2 that we cannot buy (14 in this case).

So, by finding the *smallest* numbers we can buy for each remainder, 1 to 5, taking the *largest* of these and subtracting 6 we will get the largest number of nuggets we cannot buy.

To do this we first calculate all the combinations of 9v+20w with both v and w taking on each remainder from 0 to 5. This gives us the following 6x6 table of values:

0	20	40	60	80	100
9	29	49	69	89	109
18	38	58	78	98	118
27	47	67	87	107	127
36	56	76	96	116	136
45	65	85	105	125	145

Next we calculate the remainders, modulo 6, of each of these numbers. We display these in another table, where each remainder is in the same position as that of its parent in the previous table:

0	2	4	0	2	4
3	5	1	3	5	1
0	2	4	0	2	4
3	5	1	3	5	1
0	2	4	0	2	4
3	5	1	3	5	1

So, there are six numbers in the first table that correspond to the number 2 in the second table. These are 20, 38, 56, 80, 98 and 116. The smallest of these is 20, as we noted previously. Repeating this process for each remainder 1 to 5 in turn we find:

Remainder:	1	2	3	4	5
Smallest value of 9v+20w:	49	20	9	40	29

The largest of these "smallest" numbers is 49. Hence the largest number of chicken nuggets we *cannot* buy is 49-6, or 43.