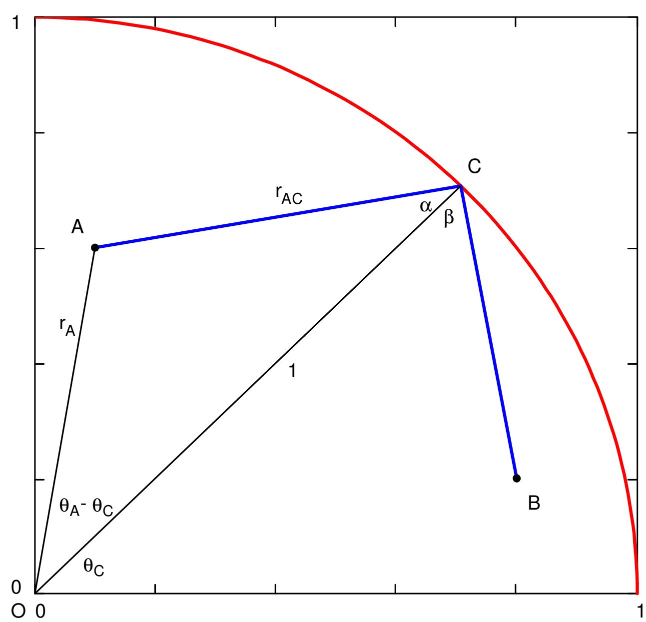
Circular Billiards

Question

Imagine a circular billiard table of unit radius, with centre at (x,y) coordinates (0,0). It has two billiard balls on it. Ball A is at coordinates (-0.8, -0.1); ball B at (0.6, 0.4). What are the coordinates of the point(s) on the cushion to which one must hit ball A so that it bounces off the cushion once and then hits ball B?

Solution

Let's first derive the condition that must be satisfied for general positions of the balls and then insert the particular positions specified in the question. Look at the illustration below, where, for clarity, we show only the first quadrant of the circle.



For the ball to go from A to B via C the angle of incidence, α, at C must equal the angle of reflection β. We must therefore obtain expressions for α and β in terms of parameters we know, and then find where C must be in order for them to have the same magnitude.

We'll call the angle between the x-axis and the radial line from the origin to C, θ_C, and the angle between the x-axis and the line from the origin to the ball at A, θ_A. Applying the sin rule to triangle OAC we have (refer to the diagram):

$$\frac{\sin\alpha}{r_A} = \frac{\sin\left(\theta_A - \theta_C\right)}{r_{AC}} \qquad \ldots(1)$$

If the Cartesian coordinates at A have the known values (x_A, y_A) then we know that $r_A = \sqrt{x_A^2 + y_A^2}$.

If the coordinates at C have the as yet unknown values (x_C, y_C), then we know that

$x_C = \cos\theta_C,\ y_C = \sin\theta_C$, so $r_{AC}^2 = \left(\cos\theta_C - r_A\cos\theta_A\right)^2 + \left(\sin\theta_C - r_A\sin\theta_A\right)^2$. This can be simplified to:

$$r_{AC}^2 = 1 + r_A^2 - 2r_A\cos\left(\theta_A - \theta_C\right)$$

Using this in equation (1) above and rearranging we obtain:

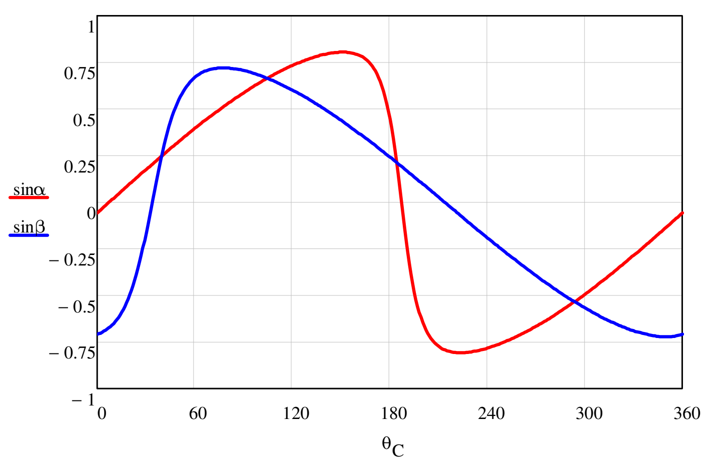
$$\sin\alpha = \frac{r_A\sin\left(\theta_A - \theta_C\right)}{\sqrt{1 + r_A^2 - 2r_A\cos\left(\theta_A - \theta_C\right)}}$$

Following a similar procedure for ball B, we find:

$$\sin\beta = \frac{r_B\sin\left(\theta_C - \theta_B\right)}{\sqrt{1 + r_B^2 - 2r_B\cos\left(\theta_C - \theta_B\right)}}$$

We therefore need to find values of θ_C that make these last two expressions equal in magnitude (if α is the same magnitude as β then $\sin\alpha$ will be the same magnitude as $\sin\beta$). Once we know θ_C we can find the corresponding Cartesian coordinates of C as ($\cos\theta_C$, $\sin\theta_C$).

Because of their form these expressions don't allow us to find a simple, analytical solution for θ_C. We can, however, use a numerical or graphical method. Let's plot $\sin\alpha$ and $\sin\beta$ against θ_C, using the values for the positions of A and B as specified in the question.



We can see there are *four* solutions. These occur at the following approximate values of θ_C : 184.1°, 39.5°, 104.5° and 293.5°. So the four possible positions of C in Cartesian coordinate form are:

(-0.997, -0.071), (0.772, 0.636), (-0.250, 0.968) and (0.399, -0.917).

The individual trajectories can be seen in the diagrams below.