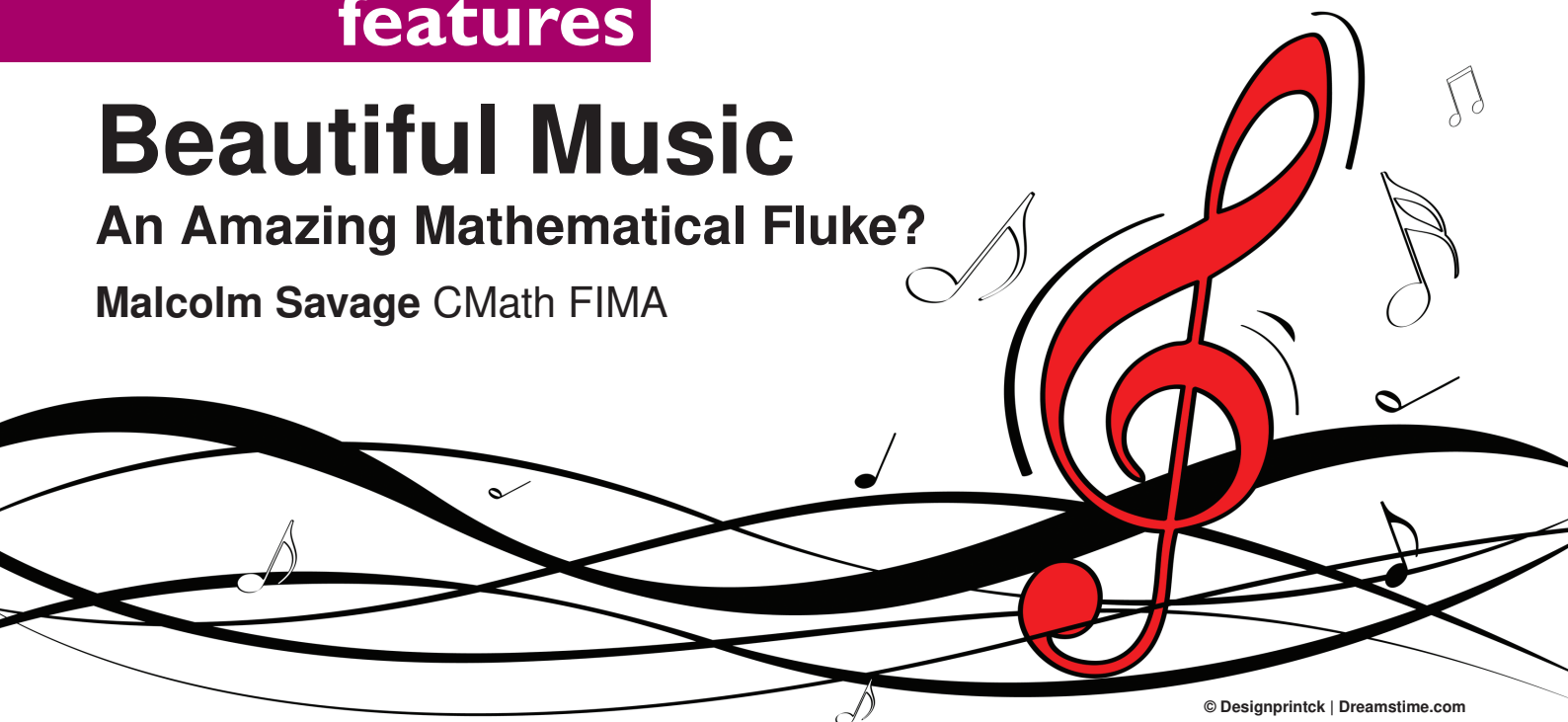


Beautiful Music

An Amazing Mathematical Fluke?

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Music has charms to soothe a savage breast,
To soften rocks, or bend a knotted oak.
By magic numbers and persuasive sound.

William Congreve, 1697

Prelude

Beautiful music is not easy to define. Some would claim that it is impossible to define because the terms ‘beautiful’ and to some extent ‘music’ are entirely subjective – the aural equivalent of ‘in the eye of the beholder’ (presumably ‘in the ear of the listener’, although the phrase doesn’t have quite such a ring to it). Such people might contend that anyone who finds the sound of rush-hour traffic more beautiful musically than, say, Mozart’s *Elvira Madigan* Concerto is taking a perfectly reasonable position.

However, whilst not attempting a complete definition of ‘beautiful music’, for the purposes of this article I postulate that ‘beautiful’ music and, for that matter, less beautiful but ‘interesting’ music, must incorporate *harmony*, *modulation* and a modicum of *complexity*.

Music, in its many forms, is a significant part of human life, from nursery rhymes to Bach. In this article, I will explain why it was quite improbable that the fundamental properties of rational and irrational numbers allowed music to exist at all.

First movement

Let us consider scales, i.e. sets of frequencies, with appropriate properties. The requirement for basic harmony dictates that for an arbitrarily chosen playable* note of frequency f , notes of frequency at or very near

$$2^n \cdot f$$

(for a range of positive and negative integers n) should also be included in the set of selectable, or playable, notes. Without pre-empting the number of notes in a scale, I will for convenience use a familiar term and refer to this as the ‘octave’ criterion.

* The term ‘playable’ means playable on some suitable instrument(s). In practice, for specific instrument/voice ranges, the range of n , m and p may vary.

A further fundamental harmony requirement demands that for an arbitrarily chosen playable note of frequency g , notes of frequency at or very near

$$\left(\frac{3}{2}\right)^m \cdot g$$

(for a range of positive and negative integers m) should also be included in the set of playable notes. For convenience I will refer to this as the ‘perfect fifth’ criterion.

Ideally it would also be the case that for an arbitrarily chosen playable note of frequency h , notes of frequency at or very near

$$\left(\frac{5}{4}\right)^p \cdot h$$

(for a range of positive and negative integers p) should also be included in the set of playable notes. For convenience I will refer to this as the ‘perfect third’ criterion.

We now consider the requirement for modulation. Modulation can be used to describe either a complete change of key or a brief excursion into, or hint of, another key. In all cases the harmony criteria must continue to apply. It is quickly evident that the only solution which also meets the ‘octave’ and ‘perfect fifth’ criteria is the so-called ‘even-tempered’ scale of, say, k basic intervals. A basic interval is that between any two adjacent playable notes the ratio of whose frequencies is constant, namely

$$2^{(1/k)}.$$

In the traditional Western scale $k = 12$ and the basic intervals are known as semitones.

To meet the ‘perfect fifth’ criterion, it must also be the case that there exists an integer j such that

$$\frac{j}{k} \approx \alpha$$

where α is defined by

$$2^\alpha = \frac{3}{2}$$

so that $\alpha \approx 0.585$.

Finally, to meet the ‘modicum of complexity’ requirement, we wish j and k to be not too small but not unduly large.

If $\sqrt{2}$ had been close enough to 1.5 (giving $j = 1$ and $k = 2$) we would have a 2-note scale providing extremely boring music. Beethoven would have been a window cleaner, and Paul McCartney would have been a double glazing salesman. Equally if the smallest pair (j, k) giving a good approximation had proved to be, say, $(17, 29)$, instrument construction and tuning would have been difficult, if not impossible, and human ability to play such instruments and/or comprehend the resulting sound would have been in serious doubt. (The basic interval between successive notes in this case would be less than a quarter of a (Western) tone. See also Appendix A.)

We now consider from first principles those candidate pairs (j, k) which could satisfy the ‘perfect fifth’ criterion. We note that

$$2^{(\alpha+1)} = 3$$

and hence

$$\alpha = \log_2 3 - 1.$$

What is the probability that the irrational $\log_2 3 - 1$ can be approximated by a rational fraction of the form j/k accurate to, say, within $\pm \frac{1}{3}\%$, with, say, $k \leq 16$?

This is quite a profound question. There is no obvious number theory or other basis that, shall we say, obliges $\log_2 3$ to have this property. Yet if it did not, music as we know it, and certainly beautiful music as described above, would not exist. This has some far-reaching implications.

Intermezzo

I acknowledge at this point that my use of the term ‘probability’ may be open to debate, since $\log_2 3$ is a constant rather than a variable. Nevertheless, I think ‘probability’ is the best term to describe the concept. For example, it would seem reasonable to discuss the probability that, say, the millionth decimal digit (after the decimal point) in the decimal representation of π , is a zero, and to feel comfortable with a conclusion that the probability is 0.1.

A more extensive philosophical discussion on the validity of the term ‘probability’ in such cases is *probably* a suitable subject for another article, but for now I propose to use my wider definition with impunity. (Incidentally, the millionth decimal digit of π is actually 1.)

Second movement

The proposed accuracy of $\pm \frac{1}{3}\%$ above derives from an informed judgement about the ability of a human ear to distinguish slight variations in frequency. We can derive the corresponding maximum error allowable in the frequency ratio $2^{(j/k)}$ as follows.

For a fraction j/k with a relative error of x in approximating to α we have

$$\frac{j}{k} = \alpha(1 + x)$$

and for the frequency ratio $2^{(j/k)}$ with a relative error of y in approximating to 1.5 we have

$$2^{(j/k)} = 1.5(1 + y).$$

This gives

$$2^{\alpha(1+x)} = 1.5(1 + y)$$

and since $2^\alpha = 1.5$ this simplifies to

$$1.5^x = 1 + y$$

and thus for $x = 0.00333 \dots$ we obtain $y \approx 0.00135$.

So the maximum percentage error for the frequency ratio $2^{(j/k)}$ is about 0.135%. This corresponds to about two musical ‘cents’. The ability of a musically discerning human ear to differentiate between perfect fifths and ‘very near misses’, due to inaccurate tuning, suggests that these error limits are appropriate. Singers and string instrument players occasionally include (typically, but not always, at the end of a piece) a ‘true fifth’ in a chord, generating a slightly smaller frequency discrepancy than the above error limits, but still with a discernible difference for players and audience.

The 19th century German mathematician Helmholtz performed various experiments to measure human perception of consonance and dissonance, which, amongst other things, did suggest that the perfect fifth is a particularly critical feature of harmony, and that a relatively slight departure from it is perceived as dissonant.

The limit $k \leq 16$ is my assessment of a practical limit for instrument construction and, perhaps more critically, the capability of the human ear to beneficially distinguish adjacent frequencies.

We can now calculate the probability that beautiful music was able to occur. The calculations which follow would change if the above accuracy or scale-size limits were varied slightly, but the overall theme would remain.

If for some particular pair of integers (j, k) the maximum allowed error occurred, the ‘target value’ of 1.5 for the frequency ratio $2^{(j/k)}$ would have been ‘missed’ by about

$$1.5 \times 0.00135 = 0.002025.$$

We now consider all ‘candidate pairs’ (j, k) with $k \leq 16$, and test whether they meet the ‘perfect fifth’ criterion. We restrict our search to within one ‘octave’ and hence $j < k$. We only include cases where j and k are co-prime. Candidate pairs such as $(3, 5)$, $(6, 10)$ and $(12, 15)$, for example, would provide scales which, although differing in the number of notes they contain, all give the same frequency ratio, $2^{0.6}$, as the co-prime pair $(3, 5)$ for the putative perfect fifth. There are a total of 79 ‘candidate pairs’:

(1, 2)
 (1, 3) (2, 3)
 (1, 4) (3, 4)
 (1, 5) (2, 5) (3, 5) (4, 5)
 ...
 (1, 16) (3, 16) ... (13, 16) (15, 16)

For each of these we can calculate the frequency ratio $2^{(j/k)}$ and then assume that the 79 values are randomly distributed over the interval $(1, 2)$. This assumption is not quite rigorous, but seems to be the best approach available. The target range for any putative perfect fifth frequency ratio is $[1.497975, 1.502025]$. This range has a span of 0.00405.

The probability that any one of the (pseudo-random) values misses the target range is

$$1 - 0.00405 = 0.99595$$

and the probability that they all miss is therefore

$$0.99595^{79} \approx 0.726$$

and the probability that at least one hits the target (i.e. that the ‘perfect fifth’ criterion is met satisfactorily) is therefore 0.274.

A similar analysis can be done for the ‘perfect third’ criterion. In this case we begin with a particular value of k which has been found to meet the ‘perfect fifth’ criterion, and consider candidate pairs (r, k) , but now including cases where r and k are not co-prime. For example, for $k = 12$ we have 11 candidate pairs. We will test for values of r for which $2^{(r/k)}$ is close to $5/4$.

Helmholtz’s work and some more recent evidence suggest that a more relaxed accuracy requirement may be appropriate for the perfect third. With a (perhaps generous) 1% maximum error, if for some particular pair of integers (r, k) this maximum allowed error occurred, the ‘target value’ of 1.25 for the frequency ratio $2^{r/k}$ would have been ‘missed’ by

$$1.25 \times 0.01 = 0.0125.$$

The target range for any putative perfect third frequency ratio is thus $[1.2375, 1.2625]$. This range has a span of 0.025. The probability that any one of the (pseudo-random) values misses the target range is therefore

$$1 - 0.025 = 0.975$$

and for $k = 12$ the probability that they all miss is therefore

$$0.975^{11} \approx 0.757$$

and the probability that at least one hits the target (i.e. that the ‘perfect third’ criterion is met satisfactorily) is therefore 0.243.

Finale

It emerges that, with $k \leq 16$, precisely one pair $(j, k) = (7, 12)$ satisfies the ‘perfect fifth’ criterion; the corresponding error in the frequency ratio is about 0.1%. In fact no further pairs come at all close to this standard until $(17, 29)$. $k = 12$ enables just the right degree of complexity, because 12 conveniently has four factors: 2, 3, 4 and 6. Moreover the 12-note scale offers the bonus that it adequately meets the ‘perfect third’ criterion, because

$$2^{(4/12)} = \sqrt[3]{2} \approx \frac{5}{4}$$

(the error is $< 0.8\%$), giving a convenient three modulations to return to the root key via perfect thirds (e.g. $A \rightarrow C\sharp \rightarrow F \rightarrow A$).

We need a more elaborate 12 modulations to return to the root key via perfect fifths:

$$A \rightarrow E \rightarrow B \rightarrow F\sharp \rightarrow C\sharp \rightarrow A\flat \rightarrow E\flat \rightarrow B\flat \\ \rightarrow F \rightarrow C \rightarrow G \rightarrow D \rightarrow A.$$

The probability of having met both the ‘perfect fifth’ and ‘perfect third’ criteria is

$$0.274 \times 0.243 \approx 0.0666.$$

So the ‘amazing mathematical fluke’ is that beautiful music has been ‘allowed to exist’ even though the probability that this would occur was only about 1 in 15.

The philosopher Nietzsche said: ‘Without music, life would be a mistake.’ I conclude by observing that:

Clearly $2^{19} \neq 3^{12}$ but fortunately $2^{19} \approx 3^{12}$.

Appendix A: Candidate pairs

The set of pairs (j, k) which give a ‘good’ (i.e. within $\pm \frac{1}{3}\%$) approximation

$$\frac{j}{k} \approx \alpha$$

makes interesting reading. Only cases where j, k are co-prime are listed. In Table 1, the % error column is the error in the frequency ratio in meeting the ‘perfect fifth’ criterion ($2^{(j/k)} \approx 1.5$).

(If we impose no limit on k we can of course achieve any degree of accuracy we wish by taking the decimal representation of α to as many decimal places as required and then expressing this as a fraction in its lowest terms.)

Table 1: Candidate pairs

j, k	% error	Notes
7, 12	0.11	This is the Western Scale. $2^{(4/12)} \approx 1.26$
17, 29	0.09	$2^{(9/29)} \approx 1.24$ (Perfect third modulations would not return to root key until 29th time.) $2^{(17/53)} \approx 1.249$
31, 53	0.004	
38, 65	0.024	
41, 70	0.052	
44, 75	0.115	
45, 77	0.038	
52, 89	0.045	
55, 94	0.01	
58, 99	0.062	
...		
179, 306	0.0003	
...		
389, 665	0.000006	$2^{(214/665)} = 1.2498929$ For a piano constructed to use this scale the player would need about 500 (very thin!) fingers, amazing ears and brain, and the piano maker would need some nifty construction techniques. Would the human race have evolved to this capability if this had been the first good approximation?

