

Mixing Motions*

Andrew D. Irving and Ebrahim L. Patel

Abstract

This paper considers the visual cycle produced by the motion of multiple objects within a bounded region. Under the influence of their frame of reference, an observer perceives a single rolling pattern. As such, the paths traced by individual objects are generally overlooked. In this piece, the identified imagery and unidentified trajectories are reconciled mathematically. Pertinent real-world systems are discussed.

Introduction

For over ten years, Arthur Shapiro¹ has devised visual phenomena to enhance our understanding of the human brain [1, 2]. When showcased in [3], one such phenomenon² became an online sensation. Based on the *Tusi couple* (depicted in Figure 1), the video depicts eight dots, which move with common frequency within a circular boundary. Each of the dots moves independently, but the effect is to make it seem as though they are all moving as a circle. Initially undetected, the reciprocating motion of individual dots never fails to surprise.⁴ Here we explore the relationship between this motion and the resulting percept through a mathematical model.



Figure 1: The Tusi couple, as sketched by Nasir al-Din al-Tusi in [4]. A fixed point on the edge of a rolling circle is marked.³

Dot model

Let us consider Shapiro's dot animation. Motion is the changing of an object's position over time. We can describe the position of each dot by a pair of Cartesian coordinates (x, y) . With respect to the circular boundary's centre (as represented by the origin in Figure 2), x and y denote a dot's horizontal and vertical displacement, respectively.

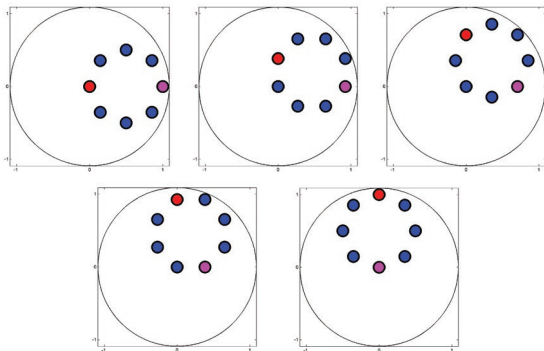


Figure 2: Simulation of [3] at $t = 0$ and moments which immediately follow (from left to right). A circle of dots appears to roll around a bigger circle.

The motion of each dot is periodic such that both x and y are sinusoidal in time [5]. Galileo showed that the simplest such waves are described by $j(t) = V \sin(2\pi ht)$, where the wave has amplitude V and frequency h in time t [6]. Hence we can employ the following pair of equations to simulate the time evolution of Shapiro's n objects,

$$\begin{aligned} x_{k,t} &= A_k \sin[2\pi ft + \mu_k] \\ y_{k,t} &= B_k \sin[2\pi gt + \nu_k] \end{aligned}$$

noting the addition of phase terms μ and ν which set the initial coordinates of dot k , where k takes the integral values in the range $[0, n - 1]$. Both phase terms as well as amplitudes A and B depend on k , while frequencies f and g do not.⁵

Optical effects

Shapiro's visual phenomenon features moving dots which collectively form a circle, c . In the outset of [3], we perceive circle c rolling along the inner edge of a bigger circle, D . As Figure 2 shows, the visual effect resembles an aerial view of a planetary mixer (e.g., see [7]). The video-maker goes on to reveal what at first seems counter-intuitive: each dot is moving in a straight line.

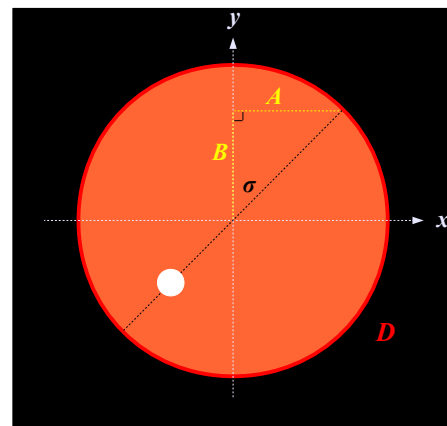


Figure 3: A white dot moves along a radial line of D . The radial line makes a clockwise angle of σ with the positive y -axis.

To replicate [3], dot k must move back and forth along some radial line of D [8]. Each such line (of length, say $2W$) makes a k -dependent angle of σ with the y -axis (as in Figure 3). Thus,

$$\begin{aligned} A_k &= W \sin \sigma_k \\ B_k &= W \cos \sigma_k \end{aligned}$$

where $\sigma = k\pi/n$ by inspection.⁶ With these terms, we can shift our focus to Shapiro's percept. As Figure 2 shows, every $n/2$ dots mark the endpoints of some radial line of c . By definition, the midpoint of each such diameter (of length, say $2w = W$) is equal to the centre of c . Hence the centre of c has coordinates (p, q) where,

$$p_t = \frac{x_{k,t} + x_{k+\frac{n}{2},t}}{2} \quad (1)$$

$$q_t = \frac{y_{k,t} + y_{k+\frac{n}{2},t}}{2} \quad (2)$$

*Early Career Mathematician Catherine Richards Prize 2015 winning article.

given a valid choice of our phase terms μ and ν . It is the relative phase of Shapiro's dots which creates c [8]. Using our expressions for (x, y) and trigonometric identities we obtain,

$$\frac{x_{k,t} + x_{k+n/2,t}}{2} = G_k \sin(2\pi ft) + H_k \cos(2\pi ft) \quad (3)$$

$$\frac{y_{k,t} + y_{k+n/2,t}}{2} = M_k \sin(2\pi gt) + N_k \cos(2\pi gt) \quad (4)$$

where G, H, M and N are functions of unknown μ and ν .⁷ Unlike the right-hand sides of (3) and (4), the left-hand sides of (1) and (2) do not vary with k . To be clear, the values of μ and ν required in (1) and (2) are precisely those which eliminate the explicit k -dependence of (3) and (4). One solution is $\mu = \nu = \sigma$.⁸ Hence,

$$p_t = w \cos(2\pi ft) \quad (5)$$

$$q_t = w \sin(2\pi gt) \quad (6)$$

while our proposed model of Shapiro's phenomenon becomes,

$$x_{k,t} = W \sin \sigma_k \sin[2\pi ft + \sigma_k] \quad (7)$$

$$y_{k,t} = W \cos \sigma_k \sin[2\pi gt + \sigma_k] \quad (8)$$

i.e., $y = \lambda x$ (where λ varies with k but not t) when $f = g$.⁹ Thus our dots oscillate along distinct diameters of D . Equations (7) and (8) describe the motion of dots with respect to the centre of D . However, with respect to the centre of c , our dots have coordinates (r, s) where,

$$r_{k,t} = W \sin \sigma_k \sin[2\pi ft + \sigma_k] - w \cos(2\pi ft) \quad (9)$$

$$s_{k,t} = W \cos \sigma_k \sin[2\pi ft + \sigma_k] - w \sin(2\pi ft) \quad (10)$$

i.e., $(r, s) = (x - p, y - q)$. By this definition $(x, y) = (r, s) + (p, q)$, i.e., the motion described by (x, y) is equivalent to the motion described by the sum of (r, s) and (p, q) . Thus we can use simplifications of (9) and (10) to obtain,

$$x_{k,t} = -w \cos[2\pi ft + 2\sigma_k] + w \cos(2\pi ft) \quad (11)$$

$$y_{k,t} = w \sin[2\pi ft + 2\sigma_k] + w \sin(2\pi ft) \quad (12)$$

i.e., $(x, y) = (-w \cos \delta + p, w \sin \delta + q)$, where δ varies with both k and t . Hence each dot makes a circular orbit around (p, q) . Using (5) and (6) it is clear that (p, q) simultaneously makes a circular orbit around $(0, 0)$.

Put simply, this means that our dots maintain a circular form that rotates on an axis which revolves around the centre of D .¹⁰ Thus viewers perceive a rotational and translational motion of c inside D , i.e., rolling [9].

In summary, our proposed model explains Shapiro's apparent paradox: linear oscillation and (non-linear) rolling can be reconciled as facets of the same motion when $\mu = \nu = \sigma$ and $f = g$.

Aggregate motion

An onlooker's perception of [3] depends on their reference frame (RF). This is illustrated by a vector form of $(x, y) = (r, s) + (p, q)$,

$$\underbrace{\vec{\text{Dot}}}_{\text{Reciprocation}} = \underbrace{\vec{\text{Dot}}_c}_{\text{Rotation}} + \underbrace{\vec{c}}_{\text{Translation}} \quad (13)$$

where $\vec{z}$ and $\vec{z}_c$ denote vectors expressed in the fixed RF (D) and in the translating RF (c), respectively. As it revolves around

D , c draws the eye along with it, i.e., a viewer automatically uses c as their RF.¹¹ Hence, as (13) suggests, the unconditioned observer perceives a spinning dot pattern.

Shapiro contrasts such imagery with the motion of individual objects. As (13) shows, the latter can be resolved into the sum of two circular motions. That is, the rectilinear movement of [3] may be viewed as the resultant of two component motions. Thus Shapiro's dots exhibit *aggregate motion* [10], a commonly engineered feature of real-world systems.

Systems of this kind achieve a desired motion indirectly, i.e., by executing simpler components. Familiar examples include planetary mixers (such as those made by Kenwood). Planetary mixing (PM) tools rotate on an axis (called a *shaft*) that revolves around a vessel's central axis.¹² In this way PM tools emulate Shapiro's dots, sharing the same basic components.

In short, (11) and (12) provide a template for PM actions. Shown in Figure 4, the coordinates (u, v) of the PM tool k

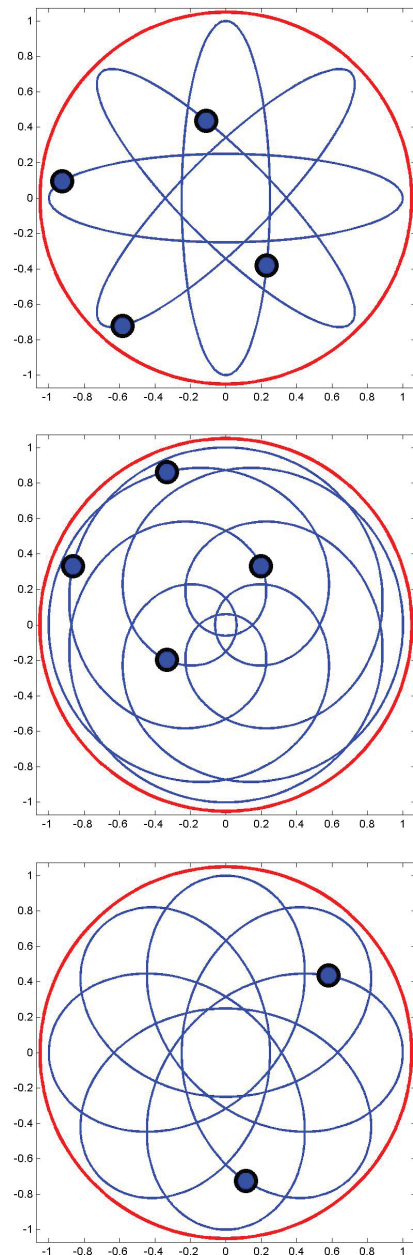


Figure 4: Cross section of vessel (red) and path of mixing tools (blue) over one revolution using Kenwood (top, bottom) and Ross (centre) mixing actions shown in [11–13].¹³

in [11–13] can be given by,

$$u_{k,t} = \tau \cos [2\pi dt + 2\sigma_k] + \epsilon \cos (2\pi bt) \quad (14)$$

$$v_{k,t} = \tau \sin [2\pi dt + 2\sigma_k] + \epsilon \sin (2\pi bt) \quad (15)$$

where the shaft's orbit has radius ϵ and frequency b while the tool's orbit has radius τ and frequency $d = \gamma b$. In Figure 4, $|\gamma| = 1, 2$ and 3 (from top to bottom). A distinguishing aspect of mixer design here, γ is a product of the component form.¹⁴ For many systems of aggregate motion, this form proves instructive.

Discussion

We have highlighted the value of abstraction: generality [14]. Composed of two circular motions, the trajectories of planetary mixing tools and Shapiro's dots can be produced using the same model, i.e. (14) and (15). Thus mathematical rules that govern Shapiro's work on cognition provide a framework for the study and improvement of mechanical design.

In [15] Shapiro's dots collectively produce a rolling effect, a spectacle also generated by a planetary mixer's tools. In contrast, these objects (i.e., dots and tools) trace aggregate paths which can be distinctive for each system (e.g., for each mixer in Figure 4). Percepts which are not easily distinguished can thus correspond to aggregate motion which is.

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Notes

1. Shapiro is a renowned psychologist at American University in Washington.
2. The original visual phenomenon can be found at [15].
3. Credit for image: Rare Books and Special Collections Division of the Library of Congress, Washington, DC. This image was presented as part of the special exhibition, *Rome Reborn: The Vatican Library & Renaissance Culture*.
4. We encourage readers to watch [3] as description alone cannot do it full justice.
5. Subscripts k and t denote explicit dependence on k and t , respectively. We omit these subscripts in the main text.
6. We inspected both [3] (where $n = 8$) and [15] (where $n = 6$). In both cases the n radial lines divide D into $2n$ sectors, each with a central angle of π/n . The dot which oscillates along the y -axis has index $k = 0$ in our model.
7. In fact,

$$G = w \sin \sigma_k \cos \mu_k + w \cos \sigma_k \cos \mu_{k+n/2};$$

$$H = w \sin \sigma_k \sin \mu_k + w \cos \sigma_k \sin \mu_{k+n/2};$$

$$M = w \cos \sigma_k \cos \nu_k - w \sin \sigma_k \cos \nu_{k+n/2};$$

$$N = w \cos \sigma_k \sin \nu_k - w \sin \sigma_k \sin \nu_{k+n/2}.$$

8. This is reflected in the findings of [16]. For the given phase terms, $G = N = 0$ while $H = M = w$.
9. For any k , points which satisfy $y = \lambda x$ (where $\lambda = \cot \sigma_k$) form a continuous straight line through $(0, 0)$. Using Pythagoras (see A and B), dots are displaced from $(0, 0)$ up to a point of contact with D .

10. For any t , points which satisfy $(x, y) = (-w \cos \delta + p, w \sin \delta + q)$ lie on a continuously defined circle of radius w , with centre (p, q) .
11. We assume that, as in [3] and [15], there are sufficient dots to allow a viewer to detect the underlying circle, c .
12. The tools used by Ross and Kenwood are blades and beater-elements, respectively. Mixing vessels are (or can be treated as) cylindrical.
13. The Ross machine in [12] has 2 shafts, each with 2 tools. Our model (which is easily modified) produces the same paths with 1 shaft and 4 tools.
14. This form describes the component, rather than resultant, motions. The value of γ in [12] is approximately, but not exactly, 2. The value of γ in [13] is -3 , although the value is -4 for the (erroneous) red path shown.

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