

# Kalman Filter Tutorial Part II – Applications

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### Kalman filter variants and extensions

#### Alpha-beta trackers

A degenerate form of the Kalman filter which does not require a priori knowledge of the state and measurement noise covariances is the alpha-beta tracker [1]. In this case, the equations are based on a (conventional) low-pass filter used to smooth out the effects of measurement noise. Using similar terminology to the Kalman filter, but with  $\mathbf{x}(t)$  explicitly used for position and  $\mathbf{v}(t)$  for velocity, the alpha-beta tracker equations are

$$\text{Position estimate:} \quad \hat{\mathbf{x}}_k = \tilde{\mathbf{x}}_k + \alpha(\mathbf{y}_k - \tilde{\mathbf{x}}_k),$$

$$\text{Velocity estimate:} \quad \hat{\mathbf{v}}_k = \hat{\mathbf{v}}_{k-1} + \frac{\beta}{T}(\mathbf{y}_k - \tilde{\mathbf{x}}_k),$$

$$\text{Position prediction:} \quad \tilde{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + T\hat{\mathbf{v}}_k.$$

These equations can be deduced from those of the Kalman filter under the following substitutions.

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \\ \beta/T & 0 \\ 0 & \beta/T \end{pmatrix}.$$

Note that the innovation in this case is just  $\mathbf{z}_k = \mathbf{y}_k - \tilde{\mathbf{x}}_k$ . The parameters  $\alpha$  and  $\beta$  control the low-pass cut-off frequency to smooth the position and velocity estimates respectively. Optimum values clearly depend on the measurement noise and the target manoeuvrability, and in this sense reflect the covariance matrices  $\mathbf{Q}$  and  $\mathbf{R}$  of the Kalman filter. Unlike the Kalman filter, however, the alpha-beta tracker equations are derived under the assumption that measurements are made at regular intervals in time  $T$ . The main advantage of the alpha-beta tracker is that the equations are simpler to implement than the Kalman filter tracker. However, the performance may not be as good as that of the Kalman filter if the noise covariances selected for the latter really are representative of the actual operational scenario.

#### Square root Kalman filter

Depending on the particular application, it may happen that the computed covariance matrices in a Kalman filter are sensitive to

round-off errors. If this is the case, then an alternative algorithm which reduces the dynamic range (and hence the sensitivity to round-off errors) is to work with the square roots of the covariance matrices. This provides an overall improvement in performance at the expense of an increase in algorithm complexity. Square root information filtering (SRIF) is a widely used technique in signal processing to reduce the dynamic range required for numerical computation.

#### Extended Kalman filter (EKF)

If either the state or the measurement equation is non-linear, then a modified form of the Kalman filter known as an extended Kalman filter (EKF) can be used. Historically, this is the earliest and simplest development of a Kalman filter for non-linear systems. It is based on linearising the equations about a target trajectory close to the actual trajectory, and basing the updating procedure on the linearised model. The EKF allows the application of Kalman filter sequential estimation procedures to non-linear systems, but the optimality of the results even under the assumption of Gaussian noise are no longer guaranteed. Indeed, it is not even guaranteed that the processing will converge to the correct solution. Hence, in cases where the target behaviour is non-linear, Kalman filter techniques must be used with caution. The same also holds true if the measurement equation is non-linear. For example, radar measures position in terms of range and bearing, which has a non-linear relationship with Cartesian position, and hence an EKF is required in this case.

#### Unscented Kalman filter (UKF)

An unscented Kalman filter (UKF) was first proposed by Julier and Uhlman in 1997 [2] and expanded upon by Wan and Van Der Merwe [3]. The authors pointed out the flaws of using an EKF and introduced an improvement. Propagation of a Gaussian random variable (GRV) through the system dynamics is a central and vital part of a Kalman filter. In the EKF the state distribution is approximated by a GRV, which is propagated analytically through the first-order linearisation of the non-linear system. This process can introduce large errors in the true posterior mean and covariance of the transformed GRV, which may then lead to

sub-optimal performance and sometimes even divergence of the filter. The UKF addresses this problem by using a deterministic sampling approach. The state distribution is again approximated by a GRV, but instead of a single sample point it is represented using a minimal set of carefully chosen sample points. These sample points more completely capture the true mean and covariance of the GRV, and when propagated through the true non-linear system, they capture the posterior mean and covariance accurately to the third order (Taylor series expansion) for any non-linearity. The EKF in contrast, only achieves first-order accuracy. Remarkably, the computational complexity of the UKF is the same order as that of the EKF.

### Particle filter (PF)

A particle filter is also known as a sequential Monte-Carlo method [4,5]. The need for such a filter arises from the fact that, to construct a truly optimum estimator for the state of a system, knowledge of the probability density function (PDF) of the estimation error is required so that the maximum likelihood estimate of the state can be obtained. For linear systems with Gaussian noise, the basic Kalman filter provides the optimum result. For the general case, an analytic closed form solution is not available. Particle filtering endeavours to provide a solution through a process of statistical sampling. In essence, at each time instant, an approximation to the PDF is obtained by propagating random samples (particles) through the system model. Note that a Kalman filter is, by definition, a particle filter – but it has only one particle. The particle filter method requires a high computational load since in general many particles are required at each sampling instant to approximate the PDF, and in the past this has hindered not only research efforts, but also inhibited its application to real world tracking problems. However, with developments in efficient re-sampling techniques, and the increased speed of digital processors, the technique can now be implemented in real time for many applications.

### Ensemble Kalman filter (EnKF)

The ensemble Kalman filter (EnKF) [6] is a recursive filter suitable for problems with a large number of state variables, such as the digitisation of partial differential equations in geophysical models. It is a Monte Carlo approximation of the Kalman filter, which avoids evolving the covariance matrix of the PDF of the state vector. Instead, the PDF is represented by an ensemble of random variables. Ideally, the ensemble variables will form a sample in which the members are not in general independent (except perhaps initially) since every time step will inevitably create dependencies. In the EnKF, however, the ensemble members

are treated as if they are both independent and Gaussian. The EnKF is obtained simply by replacing the state covariance  $\mathbf{P}$  in the Kalman gain matrix  $\mathbf{K}$  computation by the sample covariance computed from the ensemble members (ensemble covariance). The EnKF is most appropriate to use when there are a large number of variables in the Kalman filter state vector, and is an important data assimilation element of ensemble forecasting. The EnKF is related to the particle filter, in which context a particle is the same as an ensemble member, but for the EnKF all probability distributions are assumed to be Gaussian.

### Interacting multiple models (IMM)

The ability of a system to track objects and accurately predict future locations is critical to many surveillance and camera control applications. A common scenario is one in which the dynamics of the target are not known a priori, and hence it is not possible to set up an optimum filter accurately. An interacting multiple model (IMM) solution [7] is to run a number of filters in parallel, with each filter being set up with different dynamics. The output is then obtained by selecting (switching to) that filter which best represents the target location. The IMM Kalman filter provides a powerful framework for motion tracking, as it has the ability to rapidly detect and switch between motion models. This is particularly critical in some applications because the switching time between motion models is required to be extremely small. It is worth mentioning that the IMM paradigm can be applied not only to Kalman filters (KF, EKF, UKF and PF) but also to many other processes where a rapid detection of variable system dynamics is required.

### Federated Kalman filter (FKF)

A federated Kalman filter (FKF) combines data from multiple Kalman filters [8]. A particular application is in an aircraft navigation system where the data required for calculating position and velocity information is derived from a number of different sources (e.g. GPS, inertial navigation sensors, compass, altimeter, barometer, radio beacons etc.). The federated Kalman filter is able to provide performance equivalent to that of a single Kalman filter that integrates all the independent sensor data in the system. With existing sensors a single filter is often impractical, whereas the federated filter is practical. For true optimal performance it is necessary that all Kalman filters contain the same process model and make their covariance matrices available on the serial data bus. However, the federated filter can be reconfigured to provide a less optimal solution with a higher degree of fault tolerance.

A summary of the main features provided by the different types of Kalman filter is shown in Table 1.

*Table 1: Main features of different Kalman filter types*

| Filter Type        | Advantages                                           | Disadvantages                   |
|--------------------|------------------------------------------------------|---------------------------------|
| Basic Kalman       | Guaranteed optimum solution                          | Linear system, Gaussian noise   |
| Alpha-beta tracker | Simpler computation                                  | Degraded solution accuracy      |
| Square root form   | Reduced dynamic range for computation                | Square root calculation needed  |
| Extended Kalman    | Non-linear plant and/or observation equations        | Optimum solution not guaranteed |
| Unscented Kalman   | Improved performance relative to extended Kalman     | None                            |
| Particle filter    | Non-Gaussian noise, maximum likelihood solution      | High computational load         |
| Ensemble Kalman    | Simpler for large number of state variables          | Gaussian probabilities          |
| IMM                | System with variable target dynamics                 | Increased complexity            |
| Federated Kalman   | Practical for large number of different data sources | None                            |

## Applications of Kalman filters

Kalman filters are used for many different applications involving time series in which measurements are subject to random errors. The basic idea is: noisy input data is processed by the Kalman filter to provide less noisy output data. The topics described in this next section provide a representative selection of application areas where Kalman filters can often be found. Some of the applications have already been mentioned by way of illustration, and for completeness further details are provided here.

Each type of filter can often be used for more than one application, and each application can often make use of more than one type of filter. The most suitable type of Kalman filter to use with each application will depend not only on the nature of the application, but also on the required performance and the cost of implementation.

### Tracking

For historical reasons, the most well-known application of Kalman filters is perhaps the use of angle and range measurements to determine the ground tracks of moving vehicles (targets). Examples of possible targets include ships at sea as well as aircraft, drones, missiles etc. in the air. Examples of measurements are range and bearing from airport and coastal radars. If a target is transmitting radio signals, then the bearings of the transmissions from known locations is another source of data that can be used for tracking. The primary objective in many civil applications is safety – estimating and controlling target tracks to ensure that collisions are avoided. Note that when tracking targets in this way, the relationship between measurement and target track is generally non-linear, implying that an EKF or other non-linear variant must be used.

### Navigation

An application closely related to tracking, but distinct from it, is the on-board navigation (guidance) of moving vehicles. The measurements in this case can include Global Positioning System (GPS) data, as well as visual and/or radio bearing information from known locations. These measurements can be supplemented by on-board primary radar from known targets in known locations, as well as by secondary surveillance radar information from the ground. The primary objective in all navigation applications is to estimate the actual track of a vehicle relative to a well-defined reference frame. Particular devices that implement navigation algorithms based on Kalman filtering are autopilots, attitude and heading reference systems (AHRS) and inertial navigation systems (INS). Vehicles that use navigation systems include not only ships and aircraft, but also ground vehicles (e.g. cars, lorries, tanks etc.), drones, rockets and missiles.

### Satellite orbit determination

A navigation system based on GPS satellites is described in [9]. The key requirements are the accuracy of the satellite orbit (location at any given instant), and the accuracy of the satellite clock. The orbit accuracy depends on the ephemeris information; this is referenced to the World Geodetic System (WGS-84) and based on an Earth-centred Earth-fixed (ECEF) coordinate frame. The clock accuracy is derived from an on-board atomic clock synchronised to a common time reference (see the next section – Composite clock).

Precise orbit determination (POD) is achieved using a Kalman filter based on a model of satellite dynamics (Kepler's laws) in conjunction with ground-based observations (azimuth, elevation, Doppler shift, carrier phase, range etc.). Monitoring stations (MS) on the ground are necessarily in locations which have been accurately surveyed in the WGS-84 coordinate frame. The MS for GPS are located in Hawaii (north Pacific Ocean), Ascension Island (Atlantic Ocean), Diego Garcia (Indian Ocean) and Kwajalein (south Pacific Ocean), with a main control station in Colorado Springs (central USA). To obtain high accuracy, the dynamic model of satellite motion must additionally take account of external factors such as drag, tidal effects, moon gravity and solar radiation pressure. Orbit parameters need to be updated at intervals to correct for the drift in the estimated satellite location away from truth due to modelling errors and factors that cannot be accounted for. The effects of such drifts can be obtained by employing a GPS navigation algorithm and comparing the estimated locations at each MS with the accurately known surveyed locations. Corrections to the orbit parameters are then calculated and new values uploaded to the satellites. The mathematical relationships between the satellite orbit parameters and ground-based measurements are non-linear, and hence an EKF or other non-linear variant has to be used.

### Composite clock

Since there is no such thing as absolute time, all timescales are necessarily relative. An international reference standard for time (International Atomic Time TAI) is based on an ensemble of some 400 atomic clocks (caesium, rubidium, hydrogen maser etc.) from around the world. This TAI standard, however, is independent of small variations in the Earth rotation rate. The timescale used to coordinate local times around the world (Universal Coordinated Time UTC) is based on TAI with leap seconds occasionally added or subtracted to align with the Earth rotation.

For GPS it is important that the atomic clocks in each satellite are referenced to a common time standard (GPS system time). Time synchronisation is achieved via a composite clock algorithm to establish an implicit ensemble mean (IEM) of the GPS clocks. The offset of each satellite clock from the IEM then becomes a parameter of the GPS navigation message which is transmitted along with ephemeris information to users. The offset of the GPS system time from UTC can also be computed if required.

A Kalman filter can be used to compute a composite clock IEM for TAI and GPS system time. Since the clock model (offset, drift and drift rate) are linearly related to the observations, a basic linear Kalman filter is appropriate. However, due to the fact that all of the computed time offsets are relative to an unobservable IEM, there is a tendency for the error covariance to increase unbounded, which if unchecked will ultimately lead to numerical instability. A special technique is used to decompose the composite clock error covariance and remove the component arising from unobservable elements. This process ensures that the computed error covariance remains bounded, with a corresponding stable IEM.

### Computer vision

Computer vision is concerned with methods for acquiring, processing, analysing and understanding images. The purpose is, in general, to produce numerical or symbolic information from real world images. Typically, this might involve the automatic

tracking of objects or people in real time or in recorded video camera images.

Particular applications include, for example, surveillance and camera control in areas where unusual activities are required to be detected and identified. After an object of interest is identified in a particular image, a Kalman filter can be used to track this object through subsequent images in the video sequence. Problems that can arise include: allowing for changes in orientation of the object over time; changes in light conditions; changes in the trajectory of the object (direction and speed); occlusions that hinder the visual tracking of the object; and clutter (noise) in the images. An IMM, EKF or other non-linear variant is appropriate in this case, since the dynamics may not be known a priori and the trajectory and the observations may not be linearly related.

Another topic in computer vision is image restoration. The aim is to remove noise (sensor noise, motion blur etc.) from a sequence of images. Using a Kalman filter, the ‘plant’ will be a model of the local image structures, which distinguish them from the noise. After analysing the image data in terms of the local image structures, such as lines or edges, the actual images are then regarded as ‘observations’. Running the Kalman filter can then produce a cleaned-up representation of the structures in the original images.

Simultaneous localisation and mapping

Simultaneous localisation and mapping (SLAM) is a technique with a dual objective. Consider the case where someone with a handheld camera walks in and around an environment containing stationary objects, panning the camera in order to provide a record of what the space contains. In monocular SLAM the single camera images are analysed to determine (a) the 6D pose of the camera in the space (localisation i.e. 3D position and 3D orientation) and (b) the 3D structure of the space (mapping).

For this scenario, the ‘plant’ model consists of the objects in the environment whose initial positions are unknown, and the ‘observations’ are the objects detected in the sequence of camera images. Either an EKF or an UKF can then be applied to deduce the camera trajectory and the locations of the stationary objects.

In principle, multiple cameras could be used in conjunction with a more complicated SLAM algorithm. This would merge

the multiple sets of camera image sequences, and the Kalman filter would determine the multiple camera trajectories along with a 3D map of the space.

A topic closely related to SLAM is the 6D pose of a robot arm used in manufacturing. Consider, for example, a robot being used to insert components into a product. The robot arm may have a camera so that the exact location for each component relative to the arm can be identified. An EKF can be used to help manoeuvre the arm into the exact position needed for each component to be inserted into the product.

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Other application areas

The above are just a small selection of applications which use Kalman filters in some form. Some of the other areas in which Kalman filters are used include: seismology (to determine the sources of earthquakes [10]); weather forecasting (to predict the movement of weather features [11]); speech enhancement (similar to image restoration, but in 1D [12]); electrical power distribution (to find the location of line faults in a network [13]); communications (to identify channel parameters [14]); and economics, in particular macroeconomics and econometrics (to model the time evolution of economic indices such as GDP [15], stock market volatility, wages, currency exchange rates, and demand for international reserves).

Summary

Historically, it is perhaps true to say that each type of Kalman filter (basic, extension or variant) was originally developed to improve the performance for a particular application, but it often later transpired that the resulting development could then be equally applied to other applications. Hence in reality which particular type or variant is used for a given application will depend on a number of factors. These will often reduce to a trade-off between available processing power and the required performance (e.g. accuracy), but other factors (e.g. reliability) will also play a part. There is thus no unique relationship between an application and which type of Kalman filter to use. This can be seen in Table 2, which is an attempt to show the possible types of Kalman filter that may be used for each of the applications mentioned in this introductory article.

Table 2: Applications and types of Kalman filter

|                      | Linear equations |                    | Non-linear equations |     |    | Multiple filters |     |     | All |
|----------------------|------------------|--------------------|----------------------|-----|----|------------------|-----|-----|-----|
|                      | Basic            | $\alpha$ - $\beta$ | EKF                  | UKF | PF | EnKF             | IMM | FKF |     |
| Target tracking      | •                | •                  | •                    | •   | •  | •                | •   | •   | •   |
| Vehicle navigation   | •                | •                  | •                    | •   | •  | •                | •   | •   | •   |
| Orbit determination  |                  |                    | •                    | •   | •  |                  |     |     | •   |
| Composite clock      | •                |                    |                      |     |    | •                |     | •   | •   |
| Computer vision      |                  | •                  | •                    | •   | •  |                  | •   |     | •   |
| SLAM                 |                  |                    | •                    | •   | •  | •                |     |     | •   |
| Geophysics           | •                |                    | •                    | •   | •  | •                |     | •   | •   |
| Communications       |                  |                    | •                    | •   |    | •                | •   |     | •   |
| Electricity networks | •                |                    | •                    | •   |    | •                | •   |     | •   |
| Speech enhancement   | •                |                    |                      |     |    |                  |     | •   | •   |
| Econometrics         | •                |                    | •                    |     |    |                  |     | •   | •   |

## Conclusion

In this two part article a tutorial on the basic Kalman filter has been provided, along with a brief overview of some of its many extensions, variations and applications. Part 1 [16] introduces the basic Kalman filter, and appendixes containing detailed mathematical derivations are available online at <http://tinyurl.com/Kalman-appendices>. This part provides a brief overview of some of the many extensions, variations and applications of Kalman filters. Each type of Kalman estimator/filter has generally been developed for different applications, each with its own particular characteristics, requirements and limitations. The technical literature is awash with papers that describe different combinations of the many different types of Kalman filter. The interested reader may well come across many other types of estimator/filter that could legitimately be classed as being of the Kalman type. As an example, in his book [17], Simon Haykin describes how extended and unscented Kalman filters can be used for the training and use of neural networks. It is interesting to note that whilst writing this article and putting ‘Kalman filter’ into Google, it produced ‘about 878,000 results’. This must surely confirm the very valuable and ubiquitous nature of what the Hungarian born Rudolf Kálmán initiated when he wrote his 1960 paper [18]. I take my hat off to him!

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