

View from the Pennines: Ballistic Penetration

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The moors are studded with dry stone walls which enclose long abandoned fields. These untended walls have taken on the appearance of a jaw with dental problems of nightmare proportions. The remains form a series of tooth-like stumps separated by low lines where the wall has collapsed. This process has been driven by a number of small stone-falls, each created by individual circumstances. The analysis of any single minor fall is nigh on impossible; it may have been an abrupt change of wind direction or speed, the flight of a crow or any number of small perturbations which the wall had hitherto withstood. Ultimately, though, one small effect was enough to dislodge some stones.

The straw that broke the camel's back is one way of talking about ordinary life versions of the intermediate value theorem. In science terms like *catastrophe*, *fractal basin boundary* and *tippling point* are used to give this effect greater gravitas. Ballistics provides a simple example of the physical intermediate value theorem: throw a light piece of wood at an apple and it is likely to bounce off. Fire it from a crossbow and it will pass through the apple. Somewhere in between there is a critical velocity and angle of attack which forms the boundary between the bounce and the initial penetration of the skin. But this will also depend upon the fine details of the particular apple and whose head it is resting on.

Figure 1 shows a schematic diagram of the initial velocity, V_i , of a projectile against its exit or residual velocity, V_r , after penetrating a thick plate. If V_i is too low then the projectile either does not pierce the plate, or pierces but does not exit. Above a critical velocity, V_E , the exit velocity appears to increase as the square root of $V_i - V_E$.

It is very easy to provide a simple explanation of this observation. To pierce the plate takes some energy, E_1 say, to travel through the plate takes a further E_2 , and to exit the plate takes a further amount of energy, E_3 . Thus if the initial energy of the projectile is less than $E = E_1 + E_2 + E_3$ then the projectile will not penetrate through the plate, whilst if $\frac{1}{2}MV_i^2 > E$, where M is the mass of the projectile, then the projectile will exit with speed V_r given by

$$\frac{1}{2}MV_i^2 - E = \frac{1}{2}MV_r^2 \quad (1)$$

If we define an effective speed V_E by $\frac{1}{2}MV_E^2 = E$ then (1) becomes

$$V_r = \sqrt{V_i^2 - V_E^2} \quad (2)$$

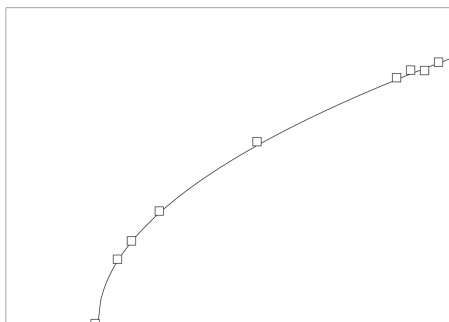


Figure 1. Schematic diagram of ballistic performance. The solid line represents the theoretical prediction of Chen and Li [2] and squares denote data points from [6] (adapted from [2]).

To see how this leads to the square root behaviour, set $v = V_i - V_E$ and note that $V_i^2 - V_E^2 = (V_i + V_E)(V_i - V_E) = (2V_E + v)v$ and so (2) is

$$V_r = \sqrt{2V_E v} \left(1 + \frac{1}{4V_E} v + \dots \right) \quad (3)$$

using the binomial theorem and ignoring quadratic terms and higher.

Although this threshold behaviour is interesting, researchers trying to understand the details of perforation need to understand much more of the interaction between the projectile and the target. A number of approaches can be taken to introduce more realism to the model [1], and we will follow Li, Chen and coworkers [2, 3, 5]. A full description would involve the very complicated interactions between a malleable projectile and the plate through which it passes. To obtain simple models with some predictive power the different phases of the motion suggested above need to be made more precise and modelled separately.

To model the initial penetration phase of the projectile, a standard approach (concentrating only on the forces opposing the motion of the projectile which is deemed to be rigid) is to assume that the opposing force is of the form

$$F_x = \alpha\sigma_y + \beta\rho V^2 \quad (4)$$

where σ_y is the yielding stress of the plate and ρ is its density. The constants α and β are explicit terms depending on the shape of the projectile. The first term is therefore a sort of bulk opposition to motion, and the second reminiscent of the Stokes drag in fluid dynamics created by the relative motion of the projectile and the target. In the limit of a very thin projectile (think about William Tell) this is the only force which needs to be taken into account, and assuming that the length of the projectile is small compared to the width H of the target so that the exiting phase of the projectile can be ignored we have

$$M \frac{dV}{dt} = -F_x \quad (5)$$

or

$$MV \frac{dV}{dx} = -(\alpha\sigma_y + \beta\rho V^2) \quad (6)$$

where we have made the distance travelled into the independent variable rather than time. This is easily integrated from V_i to V in speed and from 0 to h in space:

$$\frac{M}{2\beta} (\ln(\alpha\sigma_y + \beta\rho V_i^2) - \ln(\alpha\sigma_y + \beta\rho V^2)) = -h \quad (7)$$

After a little manipulation we find that the projectile can emerge with $h = H$ and $V = V_r$ provided $V_i > V_E$ where

$$V_E^2 = \frac{\alpha\sigma_y}{\beta\rho} \left(\exp \frac{2\beta\rho H}{M} - 1 \right) \quad (8)$$

and the residual velocity satisfies (2).

For a more bulky projectile, more physics needs to be introduced. The initial phase of penetration is as described before, but now there is a critical depth at which a solid central ‘plug’ of material is pushed ahead of the projectile. At this stage the yielding stress and Stokes drag become irrelevant, and the force on the projectile and plug together is due to shear resistance on the sides: effectively a constant resistive force, F_s , which is a constant times the total length of the plug, i.e. the effective length of contact of the plug with the target. So

$$F_s = (H^* + h)Q \quad (9)$$

where H^* is the distance from the nose of the projectile to the back of the target when the plug is formed, and h is the length of the projectile nose cone (so $H^* + h$ is the length of the plug) and Q is a constant proportional to the yield shear stress. The plug is formed when $F_s = F_x$, and this determines H^* . Motion under a constant force is of course even easier than the force in (4), so this is easily solved giving further expressions for the projectile’s residual velocity.

These arguments, given in more detail by Chen and Li [2], are still only approximations, providing heuristic expressions which appear to model some situations in which projectiles move through targets. They do not begin to describe the complications of the behaviour of the target material or the projectile during the process of puncturing the material; this would need a description in terms of continuous media which would take the

mathematics to a higher level of sophistication. Nonetheless, they bring to life different elements of elementary projectile analysis, gluing different simple approximations together, in an interesting way. □

Acknowledgement:

The idea for this article came from conversations with Steve Reid whilst I was writing about Euler’s work on ballistics [4]

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