Constructing pandiagonal magic squares of arbitrarily large size

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I first met Dame Kathleen Ollerenshaw when I had the pleasure of interviewing her in 2002 for Mathematics Today. Now 93 years old, her love of mathematics has not diminished and neither has her clarity of thought. However, problems with her eyesight have made the process of writing a labour of love. She told me that she regards this article as her 'swan song' but I am not convinced. Mathematics will not relinquish such an active mind that readily.

TERRY EDWARDS

Part 1

Magic squares have fascinated mathematicians for thousands of years. Without some guidelines it is not easy to write down even a 4×4 magic square at sight. Here I describe a method of creating arbitrarily large magic squares of a particular type called 'pandiagonal' using nothing more sophisticated than the knight's moves of the game of chess. The method is not new, but it is not generally known even to mathematicians if they have no special knowledge of magic squares. The popularity of the Su Doku number-placing puzzle may kindle new interest in magic squares, although these are not puzzles, but form a serious part of mathematical number theory known as Concrete Mathematics.

Definitions

A 'normal' magic square of order n is an array of the $n \times n$ different positive consecutive natural numbers 1 to n^2 (or 0 to n^2-1) in which the numbers in each row, in each column and in each of the two principal diagonals (running from a top corner to the diagonally opposite bottom corner) all add to the same total, namely $n(n^2+1)/2$ (or $n(n^2-1)/2$) called the **magic constant**. If, in addition, the 'broken diagonals' also add to this same magic constant the square is said to be **pandiagonal magic.**

A **broken diagonal** can be typified as starting at any position not lying on the principal diagonal through the top left corner and running diagonally downward to the right. When the diagonal reaches an edge of the square it 'wraps round' (as illustrated in Figure 1 with $n = 5$) to the opposite edge and continues diagonally downward (to the right) until reaching its starting point. Note that the broken diagonal starting at the top-right corner (here occupied by the digit 4) wraps round immediately to the left-hand column in the second row, continuing diagonally downward to the bottom row where it wraps round to its starting point in the top right corner.

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Figure 1. The principal diagonal and the four broken diagonals of a 5×5 pandiagonal magic square that run diagonally downward to the right illustrated by the positions of the digit 0 and by the digits 1, 2, 3, 4, respectively.

Paths

A diagonal is an example of a **path,** as is any row or column. We write $[r, c]$ to define a path through the square where each 'move' (or 'step') from one position in the square to the next position is r rows down and c columns across to the right. When a path reaches an edge of the square it wraps round to the opposite edge in the same way as described for broken diagonals. After n steps a path returns to its starting point.

The method of constructing pandiagonal magic squares now described is based on the paths [2, 1] and [1, 2] which are chess knight's moves. They are referred to as **knight's paths**. In chess a knight jumps at each move to a position that is not in the same row or the same column, nor in the same diagonal, a property of particular significance in creating these magic squares.

About pandiagonal magic squares

There can be no magic square of order $n = 2$, and there is just one (together with its reflections) of order $n = 3$. This is known as the *Lo-shu*, exemplified and shown in Figure 2 as a 1 to 9 square and its equivalent 0 to 8 square, which have the magic constants 15 and 12 respectively. The *Lo-shu* is not pandiagonal; the broken diagonals 6, 3, 9 and 5, 2, 8, for example, do not add to the magic constants 15 and 12.

8	1	6	or	7	0	5
3	5	7		2	4	6
4	9	2		3	8	1

Figure 2.

It was proved over a hundred years ago that there can be no pandiagonal magic squares when n is 'singly even', that is a multiple of 2 but not of 4. Pandiagonal magic squares exist for all values of n that are doubly even, that is multiples of 4, and for all odd values of $n > 3$. The number of magic squares of order 4 has been known for 350 years to be 7040, of which 384 are pandiagonal. A formula has been found for the total of a particular type of pandiagonal magic squares with special properties where n is a multiple of 4 called 'most-perfect'. When $n = 5$ the number of all 'regular' pandiagonal magic squares can be calculated from the methods of construction given here to be $2 \times (5!)^2 =$ 28,800. Enumeration in general is very difficult, and for values of $n > 5$ has so far not been achieved. Even for $n = 5$ the total, which is huge, was found only relatively recently by computer. Discussion is therefore usually restricted to pandiagonal magic squares and here, in Part 1, to squares where n is a prime number.

Here I show how to construct a pandiagonal magic square for any prime value of n (>3), however large, merely by using the two knight's paths [2, 1] and [1, 2]. The method depends on using the consecutive integers 0 to $n^2 - 1$ to form the squares and counting in base n, where each of the numbers in the squares is expressed by two digits which are exactly all the n^2 different 'ordered pairs', namely 00 to $(n-1)(n-1)$. The method also involves the use of 'Latin squares', which are the basis of many mathematical conundrums. Part I ends with a Su Doku puzzle solution that, uncharacteristically, forms a 'magic Latin square', and hence to a magic square with Su Doku characteristics.

In Part 2, I shall show how pandiagonal magic squares can be constructed by combining any two pandiagonal magic squares of, say, orders n_1 and n_2 to give a pandiagonal magic square of order $n = n_1 \times n_2$, thus enabling the creation of pandiagonal magic squares of arbitrarily large size.

Auxiliary and Latin squares

As working examples, a 4×4 and a 5×5 pandiagonal magic square are shown in Figures 3 and 4, first in the decimal system and then, respectively, in base 4 and base 5. The bold numbers illustrate a broken diagonal.

0	**11**	6	13	,	00	**23**	12	31
14	5	**8**	3		32	11	**20**	03
9	2	15	**4**		21	02	33	**10**
7	12	1	10		**13**	30	01	22

Figure 3. A pandiagonal magic 4×4 square in decimal and in base 4.

0	**14**	23	7	16	,	00	**24**	43	12	31
22	6	**15**	4	13		42	11	**30**	04	23
19	3	12	**21**	5		34	03	22	**41**	10
11	20	9	18	**2**		21	40	14	33	**02**
8	17	1	10	24		**03**	32	01	20	44

Figure 4. A pandiagonal magic 5×5 square in decimal and in base 5.

0	2	4	1	3	0	4	3	2	1
4	1	3	0	2	2	1	0	4	3
3	0	2	4	1	4	3	2	1	0
2	4	1	3	0	1	0	4	3	2
1	3	0	2	4	3	2	1	0	4

Figure 5. A pair of orthogonal 5×5 auxiliary Latin pandiagonal magic squares, which are reflections of one another in a principal diagonal.

Suppose now that, in the 5×5 (base 5) square in Figure 4, we separate the left-hand and the right-hand digits and write them as the two separate squares shown in Figure 5. They are called the **radix** and the **unit auxiliary squares**, respectively. They are not themselves magic because they do not contain 5×5 different numbers: each contains the five digits 0, 1, 2, 3, 4 five times. They are known as **Latin squares**.

An $n \times n$ **Latin square** is defined as an array of n different symbols, here the numbers $0, 1, 2, \ldots, n-1$, arranged in such a way that each symbol occurs once and once only in each row and in each column. When, in a Latin square, the different symbols also occur once and only once in each of the two principal diagonals, it is called a **Latin magic square**. When, in addition, the broken diagonals all also contain each symbol once and only once it is described as **Latin pandiagonal magic**. Two different 0 to $(n-1)$ Latin squares of order n are said to be **orthogonal** if, when combined to form the radix and unit components of a square, each of the different n^2 ordered pairs of digits occurs once and only once to form the square. For example, the two single-digit 5×5 squares shown above are orthogonal Latin pandiagonals. The individual digits 0, 1, 2, 3, 4 lie on the paths [2, 1] and [1, 2] respectively, and the two single-digit squares are reflections of one another in their principal diagonal through the top left corners. They are orthogonal and combine to form the pandiagonal magic square shown.

Constructing a pandiagonal magic square by means of knight's paths

Consider again the positions of the five digits 0, 1, 2, 3, 4 in the single-digit 5×5 radix and unit Latin pandiagonal magic squares above, first with the radix auxiliary. In an empty 5×5 grid begin by placing 0 in the top left corner. Then place the other four 0s along the knight's path [2, 1], as illustrated in Figure 6.

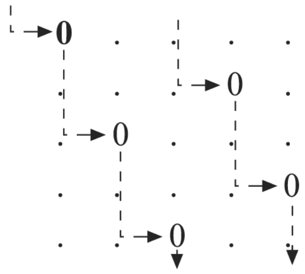

Figure 6. Positions of the 0s in the 5×5 radix auxiliary square.

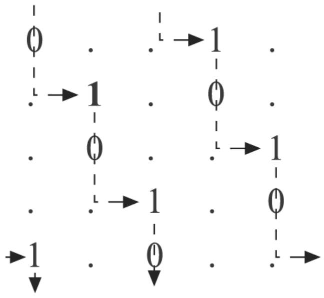

Figure 7.

Choose any one of the other digits 1, 2, 3 or 4 (say, 1) and place it in any position not occupied by a 0 (say, in the position one row down and one column across from the 0 in the top left corner). With this first 1 in position, place the other four 1s along path [2, 1] using the technique of a path wrapping round when encountering an edge. This gives the arrangement shown in Figure 7.

With $n = 5$ (or any prime number greater than 3) there will always be empty positions awaiting.

Now choose another of the numbers, say 2, and place this in any still unoccupied cell, say in the position next to the 1 down the principal diagonal. Place the other four 2s along the similar knight's path [2, 1], wrapping round when an edge is encountered. Again, there will always be vacant cells available and the five 2s will complete a path back to the first 2. Follow the same procedure with the first 3 placed (arbitrarily) next to the 2 in the principal diagonal. Put the remaining four 3s into positions that form the same [2, 1] knight's path as the other digits so far. The five 4s then fill the remaining vacant cells along a [2, 1] path. This grid now forms precisely the single-digit 5×5 radix auxiliary square shown earlier.

In the same way, create the single-digit unit pandiagonal Latin square with 0 in the top left corner by using the knight's path [1, 2]. The two squares are orthogonal reflections of one another and combine to form the square already illustrated.

This method of construction is completely general and gives a pandiagonal magic square for all prime values of $n > 3$, however large.

The first known example of a pandiagonal magic square with $n > 4$ that could have been constructed by the method of auxiliary squares and knight's paths is a 7×7 square devised by Leonhard Euler (1705–83) when seeking a solution to his '36 Officers Problem'. This square lay unnoticed for over a hundred years. It is given in the book published by the IMA in 1998 entitled *Most-perfect pandiagonal magic squares: their construction and enumeration* by myself and David Brée.

When $n > 3$ is odd but not prime the two knight's paths [2, 1] and [1, 2] 'clash' and the method described above fails. In a new Part 3 to appear in a subsequent issue, an alternative pair of knight's paths is used. The process then succeeds for all odd values of $n > 3$ – a simple interchange of numbers being required when n is also a multiple of 3. Pandiagonal magic squares of (singly) even order present totally different problems and are not discussed here.

The Su Doku puzzle and magic squares

Although not strictly anything to do with magic squares, a Su Doku number-placing puzzle solution can be Latin magic as shown below. It is probable that no Su Doku solution can be Latin pandiagonal magic, but I have made no attempt to prove this.

Su Doku solutions are all Latin squares but are not in general magic, the numbers 1 to 9 occurring once and only once in each row and in each column and in each of the nine 3×3 boxes, with no special requirements related to the diagonals. Figure 8 shows an example of a Su Doku solution in which the two principal diagonals (but not the broken diagonals) each contain one and only one of the numbers 1 to 9; it is thus a 'Latin' magic square, as defined above.

1	2	3	8	9	7	6	4	5
4	5	6	2	3	1	9	7	8
7	8	9	5	6	4	3	1	2
3	1	2	7	8	9	5	6	4
6	4	5	1	2	3	8	9	7
9	7	8	4	5	6	2	3	1
2	3	1	9	7	8	4	5	6
5	6	4	3	1	2	7	8	9
8	9	7	6	4	5	1	2	3

Figure 8. Example of a Su Doku solution that is also Latin magic.

We can think of this as a 9×9 auxiliary magic square with the numbers 0 to 8 each repeated 9 times. It would then be the square shown in Figure 9.

0	1	2	7	8	6	5	3	4
3	4	5	1	2	0	8	6	7
6	7	8	4	5	3	2	0	1
2	0	1	6	7	8	4	5	3
5	3	4	0	1	2	7	8	6
8	6	7	3	4	5	1	2	0
1	2	0	8	6	7	3	4	5
4	5	3	2	0	1	6	7	8
7	8	6	5	3	4	0	1	2

Figure 9.

1	13	25	66	78	63	47	32	44
29	41	53	10	22	7	75	60	72
57	69	81	38	50	35	19	4	16
26	2	14	61	64	76	45	48	33
54	30	42	8	11	23	70	73	58
79	55	67	36	39	51	17	20	5
15	27	3	77	62	65	31	43	46
40	52	28	24	9	12	59	71	74
68	80	56	49	34	37	6	18	21

Figure 10. A 0 to 8 magic square of order 9 (in decimals) in which, in addition to the numbers in the rows, columns and principal diagonals adding to the magic constant 369, the numbers in each of the nine 3×3 'boxes' also add to the magic constant, in the style of a solution to a Su Doku puzzle.

We can use this single-digit auxiliary square in base 9, and take it, together with its reflection in the principal diagonal running from the top-left corner downward to the right, as a single-digit radix and unit auxiliary squares to produce a (0 to 80) magic square of order 9; we can then turn this into decimals and add 1 to each number. We then have the (1 to 81) 9×9 magic square in Figure 10 in which the numbers in the nine 3×3 boxes, as well as the numbers in each row, each column and each of the two principal diagonals, all add to the magic constant 369.❑

Part 2

Here I describe a method of constructing pandiagonal magic squares of arbitrarily large size by combining squares of smaller sizes.

From a 4×4 pandiagonal magic square a ‘composite’ 16×16 pandiagonal magic square can be constructed. From this a still larger composite 256×256 pandiagonal magic square can similarly be constructed, and so on. The same method leads to the construction of ever-larger squares by combining any two pandiagonal squares of whatever order. For example, a 4×4 square can combine with a 5×5 square in two different ways to give two different 20×20 pandiagonal magic squares with the same magic constant, $20(400-1)/2 = 3990$, where the formula for the magic constant was shown in Part 1 to be $n(n^2-1)/2$.

The idea is not new. Composite squares were described by Rouse Ball before 1921 in the construction of 9×9 magic (not pandiagonal) squares, by Benson and Jacoby in 1976 and by Pickover in 2002, but using the 3×3 *Lo-shu*, which is not pandiagonal and cannot be used to produce a composite pandiagonal square. The principle of the construction is, however, general and any two pandiagonal magic squares can always be combined, as established here, to form a larger composite pandiagonal square.

The smallest composite pandiagonal magic squares are those constructed from two 4×4 squares to form 16×16 squares. The procedure is easier to follow (and more general) if two different values, say, n_1 and n_2, are used as illustration. The smallest composite squares that can be constructed from squares of two different orders are when $n_1 = 4$ and $n_2 = 5$ or vice versa. The squares used as illustration are those already encountered in Part 1 written in the decimal system, namely Figure 11.

$$\begin{array}{cccc} 0 & 14 & 3 & 13 \\ 11 & 5 & 8 & 6 \\ 12 & 2 & 15 & 1 \\ 7 & 9 & 4 & 10 \end{array} \quad \text{and} \quad \begin{array}{ccccc} 0 & 23 & 16 & 14 & 7 \\ 19 & 12 & 5 & 3 & 21 \\ 8 & 1 & 24 & 17 & 10 \\ 22 & 15 & 13 & 6 & 4 \\ 11 & 9 & 2 & 20 & 18 \end{array}$$

Figure 11. A (0 to 15) 4 × 4 pandiagonal magic square and a (0 to 24) 5 × 5 pandiagonal magic square in decimals.

Suppose the two squares that are to be combined are of order $n_1(>3)$ and $n_2(>3)$, leading to a ‘composite square’ of order $n_1 \times n_2 = n$ Denote by m_1, m_2, and m their respective magic constants, namely, $m_1 = n_1(n_1^2-1)/2, m_2 = n_2(n_2^2-1)/2$ and $m = n(n^2-1)/2 = n_1 n_2(n_1^2 n_2^2 - 1)/2$.

Constructing a composite pandiagonal magic square from two pandiagonal magic squares The base quilt

Imagine making a square patchwork quilt with n_1^2 identical patches, each patch being an $n_2 \times n_2$ pandiagonal magic square with 0 in its top left corner. To be specific, write $n_1 = 4$ ($m_1 = 30$), $n_2 = 5$ ($m_2 = 60$) and use the squares shown above. I call this a ‘4/5 base quilt’ (Figure 12). Label the 16 patches with the first 16 letters of the alphabet A to P, as shown in this and subsequent figures. Also, number the columns as $c =$ **0**, 1, 2, 3, 4; **5**, 6, 7, 8, 9; **10**, 11, 12, 13, 14; **15**, 16, 17, 18 and 19, as shown in Figure 12 and implied elsewhere. The numbers c = 0, 5, 10 and 15 are highlighted because these columns on the base quilt contain the same repetitions of the left-hand column of the $n_2 \times n_2$ squares.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

c = 0	c = 1	c = 2	c = 3	c = 4	c = 5	c = 6	c = 7	c = 8	c = 9	c = 10	c = 11	c = 12	c = 13	c = 14	c = 15	c = 16	c = 17	c = 18	c = 19
0	23	16	14	7	0	23	16	14	7	0	23	16	14	7	0	23	16	14	7
19	12	5	3	21	19	12	5	3	21	19	12	5	3	21	19	12	5	3	21
8	1	24	17	10	8	1	24	17	10	8	1	24	17	10	8	1	24	17	10
22	15	13	6	4	22	15	13	6	4	22	15	13	6	4	22	15	13	6	4
11	9	2	20	18	11	9	2	20	18	11	9	2	20	18	11	9	2	20	18
0	23	16	14	7	0	23	16	14	7	0	23	16	14	7	0	23	16	14	7
19	12	5	3	21	19	12	5	3	21	19	12	5	3	21	19	12	5	3	21
8	1	24	17	10	8	1	24	17	10	8	1	24	17	10	8	1	24	17	10
22	15	13	6	4	22	15	13	6	4	22	15	13	6	4	22	15	13	6	4
11	9	2	20	18	11	9	2	20	18	11	9	2	20	18	11	9	2	20	18
0	23	16	14	7	0	23	16	14	7	0	23	16	14	7	0	23	16	14	7
19	12	5	3	21	19	12	5	3	21	19	12	5	3	21	19	12	5	3	21
8	1	24	17	10	8	1	24	17	10	8	1	24	17	10	8	1	24	17	10
22	15	13	6	4	22	15	13	6	4	22	15	13	6	4	22	15	13	6	4
11	9	2	20	18	11	9	2	20	18	11	9	2	20	18	11	9	2	20	18
0	23	16	14	7	0	23	16	14	7	0	23	16	14	7	0	23	16	14	7
19	12	5	3	21	19	12	5	3	21	19	12	5	3	21	19	12	5	3	21
8	1	24	17	10	8	1	24	17	10	8	1	24	17	10	8	1	24	17	10
22	15	13	6	4	22	15	13	6	4	22	15	13	6	4	22	15	13	6	4
11	9	2	20	18	11	9	2	20	18	11	9	2	20	18	11	9	2	20	18

Figure 12. The ‘4/5 base quilt’.

The numbers on this ‘4/5 base quilt’, taken as a whole, do not form a magic square, because each of the 25 numbers 0 to 24 on the individual patches occurs 16 times overall. But nonetheless they have some of the same characteristics: the sum of the $4 \times 5 = 20$ numbers in each row, in each column and in each diagonal (the broken diagonals as well as the two principal diagonals) all add to the same total, namely 4×60; that is, four times the magic constant of the 5×5 squares on the individual patches, namely $n_1 m_2$. This is plain for the rows, columns and principal diagonals, but more explanation (and proof) is required to establish that the numbers in the broken diagonals also add to the required total $n_1 m_2$.

The three stages of the proof

The method of construction and the proof of its validity are demonstrated in three stages. The first stage is to establish that, in the base quilt made of n_1^2 patches each depicting identical $n_2 \times n_2$ pandiagonal magic squares, the numbers in the broken diagonals (as well as those in the rows, columns and principal diagonals) add to the same total $n_1 m_2$.

The second stage is to adjust the numbers on the original individual patches so that each number from 0 to $(n_1^2 n_2^2 - 1)$—that is, from 0 to $(16 \times 25 - 1) = 399$ when $n_1 = 4$ and $n_2 = 5$—appears once and only once on the adjusted patches taken together; and then to arrange these adjusted patches so as to form a new quilt (the ‘composite quilt’) in which the numbers as a whole form a pandiagonal magic square of order $n_1 n_2 = n = 20$.

The third stage is to establish that the construction does indeed result in a pandiagonal magic square of order $n_1 n_2 = n$ with the magic constant $m = n_1 n_2(n_1^2 n_2^2 - 1)/2 = 3990$.

Stage 1: broken diagonals in the base quilt

The principal diagonals in the base quilt that run from one top corner to the diagonally opposite bottom corner each consist of n_1 copies of the principal diagonals of the identical squares on each patch, the numbers in them summing to the required n_1m_2. The broken diagonals are not as simple to envisage.

In the identical squares on the patches forming the base quilt, as in any magic square, the broken diagonals can be read from any starting position and taken either as running diagonally downward to the right or left (the 'down' diagonals) or as running diagonally upward to the right or left (the 'up' diagonals). If they are taken as starting on the top row (at positions other than the top left corner) and are read diagonally downward to the right, they have just one break: at the right-hand edge of the square where they then wrap round (as explained in Part I) to the opposite left-hand edge and continue diagonally downward until ending at the bottom row.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

c = 0	c = 1	c = 2	c = 3	c = 4	c = 5	c = 6	c = 7	c = 8	c = 9	c = 10	c = 11	c = 12	c = 13	c = 14	c = 15	c = 16	c = 17	c = 18	c = 19
0	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7
19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21
8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10
22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4
11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**
0	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7
19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21
8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10
22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4
11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**
0	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7
19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21
8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10
22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4
11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**
0	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7	**0**	23	16	14	7
19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21	19	**12**	5	3	21
8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10	8	1	**24**	17	10
22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4	22	15	13	**6**	4
11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**	11	9	2	20	**18**

Figure 13. Base quilt plus principal diagonals at c = 0 and broken diagonals at c = (5, 10, 15).

Consider the broken diagonals in the base quilt. Figure 13 shows the broken diagonals that start in the top row at columns c = 5, 10 and 15 that run downward to the right. Figure 14 shows the 'parallel' diagonals that start in the top row at columns 2, 7, 12 and 17. The broken diagonals that start at the remaining columns are not illustrated, but have similar characteristics.

Consider first the diagonals illustrated in Figure 13 that start in the top row at columns $c = 5$, 10 and 15. The broken diagonal that starts in the top row at $c = 5$ (in patch B) consists of the principal diagonals in each of the patches B, G, L and M, and thus gives the sum n_1m_2. Likewise, the broken diagonal that starts in the top row at $c = 10$ (in patch C) consists of the principal diagonals in the patches C, H, I and N, giving the required total. So also the broken diagonal that starts at $c = 15$ (in patch D) consists of the principal diagonals in patches D, E, J and O.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

c = 0	c = 1	c = 2	c = 3	c = 4	c = 5	c = 6	c = 7	c = 8	c = 9	c = 10	c = 11	c = 12	c = 13	c = 14	c = 15	c = 16	c = 17	c = 18	c = 19
0	23	**16**	14	7	0	23	16	14	7	0	23	16	14	7	0	23	16	14	7
19	12	5	**3**	21	19	12	5	3	21	19	12	5	3	21	19	12	5	3	21
8	1	24	17	**10**	8	1	24	17	10	8	1	24	17	10	8	1	24	17	10
22	15	13	6	4	**22**	15	13	6	4	22	15	13	6	4	22	15	13	6	4
11	9	2	20	18	11	**9**	2	20	18	11	9	2	20	18	11	9	2	20	18
0	23	16	14	7	0	23	**16**	14	7	0	23	16	14	7	0	23	16	14	7
19	12	5	3	21	19	12	5	**3**	21	19	12	5	3	21	19	12	5	3	21
8	1	24	17	10	8	1	24	17	**10**	8	1	24	17	10	8	1	24	17	10
22	15	13	6	4	22	15	13	6	4	**22**	15	13	6	4	22	15	13	6	4
11	9	2	20	18	11	9	2	20	18	11	**9**	2	20	18	11	9	2	20	18
0	23	16	14	7	0	23	16	14	7	0	23	**16**	14	7	0	23	16	14	7
19	12	5	3	21	19	12	5	3	21	19	12	5	**3**	21	19	12	5	3	21
8	1	24	17	10	8	1	24	17	10	8	1	24	17	**10**	8	1	24	17	10
22	15	13	6	4	22	15	13	6	4	22	15	13	6	4	**22**	15	13	6	4
11	9	2	20	18	11	9	2	20	18	11	9	2	20	18	11	**9**	2	20	18
0	23	16	14	7	0	23	16	14	7	0	23	16	14	7	0	23	**16**	14	7
19	12	5	3	21	19	12	5	3	21	19	12	5	3	21	19	12	5	**3**	21
8	1	24	17	10	8	1	24	17	10	8	1	24	17	10	8	1	24	17	**10**
22	15	13	6	4	22	15	13	6	4	22	15	13	6	4	22	15	13	6	4
11	**9**	2	20	18	11	9	2	20	18	11	9	2	20	18	11	9	2	20	18

Figure 14. Base quilt plus diagonal at c = 2.

For the broken diagonals starting in the top row at other columns, use as illustration Figure 14 (where that starting at $c = 2$ in patch A is highlighted). This broken diagonal that starts with the 3 ($= n_2 - c$) numbers 16, 3 and 10 in patch A is followed by the 2 ($= c$) numbers 22 and 9 in patch B. This sequence of 5 ($= n_2$) numbers forms a broken diagonal of the 5×5 squares (as explained in Part I) and thus has the total $m_2 = 60$. The diagonal continues through patches F and G, and then K and L, ends in patches P and M after wrapping round, having hit the right-hand edge of the square in patch P. The sum of the numbers in this completed broken diagonal is thus $4m_2 = n_1m_2$, as required.

These sequences of the same five numbers that add to m_2 can be thought of as being divided into two 'complementary' segments (here of lengths 3 and 2), the numbers in which together add to m_2.

Notice here for reference at the next stage that the first of the segments, of lengths $3 = 5 - 2 = (n_2 - c)$ numbers, lie in the patches A, F, K and P; the second (complementary) segments of length $c = 2$ numbers lie in the patches B, G, L and M. Reference to Figures 12 and 14 shows that this pattern is exactly repeated in the (parallel) broken diagonals that start in the top row at columns $c = 7$, $c = 12$ and $c = 17$ (which can be expressed mathematically as $n - c \bmod n$, where $n = n_1n_2$; that is, here, $20 - c \bmod 20$). When $c = 7$ the two different segments lie in patches B G L M and patches C H I N, respectively. When $c = 12$ the segments lie in patches C H I N and D G J O, respectively. When $c = 17$ they lie in D G J O and A E K P, respectively. Note that the groups of four patches are analogous to the principal and broken diagonals of a 4×4 pandiagonal magic square—a fact that is central to the proof (that is to follow) that the numbers in all the diagonals of the composite square add to the required magic constant.

This completes Stage 1.

Stage 2: the 'new' composite quilt. Additions to the numbers in individual patches

The first task in creating a quilt that depicts a composite pandiagonal magic square of order n_1n_2 is to make a new set of patches that, taken together, contain each number from 0 to $(n_1n_2)^2 - 1$ once and only once. The second task is to stitch the new patches together in such a way that the numbers on the new 'composite' quilt form a pandiagonal magic square.

Without loss of generality we can specify that 0 lies in the top left corner of the composite square. Use the base quilt already described made from $n_1^2 = 4 \times 4 = 16$ patches, each depicting identical pandiagonal magic squares of order n_2 (= 5). Leave unaltered the patch that is to occupy the top left corner of the composite quilt; it contains the numbers 0 to $n_2^2 - 1$, namely 0 to 24. To each number in the first of the other patches add $1 \times n_2^2 = 25$ to make them n_2^2 to $2n_2^2 - 1$, namely 25 to 49; to each number in the next patch add $2 \times n_2^2 = 50$ to make them $2 \times n_2^2$ to $3n_2^2 - 1$, namely 50 to 74; and so on, until, in the last remaining original patch, add to each number $(n_1 - 1) \times n_2^2$ to make them $(n_1 - 1)n_2^2$ to $(n_1n_2^2 - 1)$, namely 15×25 to $16 \times 25 - 1$, or 375 to 399.

This takes care of all the numbers from 0 to $n_1^2n_2^2 - 1$, namely 0 to 399, ensuring that each occurs once and only once in the $n_1^2 = 16$ new patches taken together.

Assembling the new patches

These new patches now have to be stitched together in positions that, in the new quilt as a whole, form a pandiagonal magic square of order $n = n_1n_2$. This is achieved by placing the adjusted patches in positions in such a way that the numbers in their top left corners (where the 0s had been in the base quilt) themselves form a pandiagonal magic square of order n_1. To illustrate, let this square be that used earlier, namely Figure 15.

0	14	3	13
11	5	8	6
12	2	15	1
7	9	4	10

Figure 15.

These numbers have then to be added to each number within the each of the separate patches in matching positions, as illustrated in Figure 16. Call the square of 16 numbers on the right the 'additions square', illustrated in Figure 17.

With these additions made to the numbers in the patches already in the base quilt, the composite square of order $n_1n_2 = n$ is completed (see Figure 17).

A	B	C	D	0×25	14×25	3×25	13×25
E	F	G	H	11×25	5×25	8×25	6×25
I	J	K	L	12×25	2×25	15×25	1×25
M	N	O	P	7×25	9×25	4×25	10×25

Figure 16

Figure 18 shows the equivalent composite square with n_1 and n_2 reversed so that $n_1 = 5$ and $n_2 = 4$.

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

Addition guide

Base	Add 14×25	Add 3×25	Add 13×25
Add 11×25	Add 5×25	Add 8×25	Add 6×25
Add 12×25	Add 2×25	Add 15×25	Add 1×25
Add 7×25	Add 9×25	Add 4×25	Add 10×25

$c = 0$	$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	$c = 6$	$c = 7$	$c = 8$	$c = 9$	$c = 10$	$c = 11$	$c = 12$	$c = 13$	$c = 14$	$c = 15$	$c = 16$	$c = 17$	$c = 18$	$c = 19$
0	23	16	14	7	350	373	366	364	357	75	98	91	89	82	325	348	341	339	332
19	12	5	3	21	369	362	355	353	371	94	87	80	78	96	344	337	330	328	346
8	1	24	17	10	358	351	374	367	360	83	76	99	92	85	333	326	349	342	335
22	15	13	6	4	372	365	363	356	354	97	90	88	81	79	347	340	338	331	329
11	9	2	20	18	361	359	352	370	368	86	84	77	95	93	336	334	327	345	343
275	298	291	289	282	125	148	141	139	132	200	223	216	214	207	150	173	166	164	157
294	287	280	278	296	144	137	130	128	146	219	212	205	203	221	169	162	155	153	171
283	276	299	292	285	133	126	149	142	135	208	201	224	217	210	158	151	174	167	160
297	290	288	281	279	147	140	138	131	129	222	215	213	206	204	172	165	163	156	154
286	284	277	295	293	136	134	127	145	143	211	209	202	220	218	161	159	152	170	168
300	323	316	314	307	50	73	66	64	57	375	398	391	389	382	25	48	41	39	32
319	312	305	303	321	69	62	55	53	71	394	387	380	378	396	44	37	30	28	46
308	301	324	317	310	58	51	74	67	60	383	376	399	392	385	33	26	49	42	35
322	315	313	306	304	72	65	63	56	54	397	390	388	381	379	47	40	38	31	29
311	309	302	320	318	61	59	52	70	68	386	384	377	395	393	36	34	27	45	43
175	198	191	189	182	225	248	241	239	232	100	123	116	114	107	250	273	266	264	257
194	187	180	178	196	244	237	230	228	246	119	112	105	103	121	269	262	255	253	271
183	176	199	192	185	233	226	249	242	235	108	101	124	117	110	258	251	274	267	260
197	190	188	181	179	247	240	238	231	229	122	115	113	106	104	272	265	263	256	254
186	184	177	195	193	236	234	227	245	243	111	109	102	120	118	261	259	252	270	268

Figure 17. The 'additions square': the $n_1 = 4$, $n_2 = 5$ combined 20×20 composite square.

Stage 3: checking the composite square

To check the validity of the construction we need to establish that the numbers in the rows, columns and diagonals (the broken as well as the principal diagonals) add to the same magic constant $m = n_1n_2(n_1^2n_2^2 - 1)/2$.

In the base quilt this was straightforward, because all the patches were alike. In the composite quilt the patches have been overlaid (that is, replaced): each of the n_2^2 numbers in each of the n_1^2 patches other than the top left patch have been increased by the addition of the n_1 different multiples of n_2^2 other than 0.

Refer to Figure 17 and think of the numbers in the (named) patches of the composite quilt as the sum of the numbers in the same relative positions as in the base quilt plus the additions indicated at the top of Figure 17.

Remember that, by definition, the squares of orders n_1 and n_2 are both pandiagonal, so that the numbers in their rows, columns and all diagonals have the same sum, the magic constants m_1 and m_2. Thus the sum of the numbers in any row of the composite quilt is the sum of the numbers in the corresponding row of the base quilt, namely n_1m_2 plus the 'overlaying' additions. When $n_1 = 4$ and $n_2 = 5$, this total is

$$4 \times 60 + 5 \times 25 \times (0 + 14 + 3 + 13) = 4 \times 60 + 5^3 \times 30,$$

which, in general, is $n_1m_2 + n_2^3m_1$

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

Addition guide

Base	Add 23×16	Add 16×16	Add 14×16	Add 7×16
Add 19×16	Add 12×16	Add 5×16	Add 3×16	Add 21×16
Add 8×16	Add 1×16	Add 24×16	Add 17×16	Add 10×16
Add 22×16	Add 15×16	Add 13×16	Add 6×16	Add 4×16
Add 11×16	Add 9×16	Add 2×16	Add 20×16	Add 18×16

$c=0$	$c=1$	$c=2$	$c=3$	$c=4$	$c=5$	$c=6$	$c=7$	$c=8$	$c=9$	$c=10$	$c=11$	$c=12$	$c=13$	$c=14$	$c=15$	$c=16$	$c=17$	$c=18$	$c=19$
0	14	3	13	368	382	371	381	256	270	259	269	224	238	227	237	112	126	115	125
11	5	8	6	379	373	376	374	267	261	264	262	235	229	232	230	123	117	120	118
12	2	15	1	380	370	383	369	268	258	271	257	236	226	239	225	124	114	127	113
7	9	4	10	375	377	372	378	263	265	260	266	231	233	228	234	119	121	116	122
304	318	307	317	192	206	195	205	80	94	83	93	48	62	51	61	336	350	339	349
315	309	312	310	203	197	200	198	91	85	88	86	59	53	56	54	347	341	344	342
316	306	319	305	204	194	207	193	92	82	95	81	60	50	63	49	348	338	351	337
311	313	308	314	199	201	196	202	87	89	84	90	55	57	52	58	343	345	340	346
128	142	131	141	16	30	19	29	384	398	387	397	272	286	275	285	160	174	163	173
139	133	136	134	27	21	24	22	395	389	392	390	283	277	280	278	171	165	168	166
140	130	143	129	28	18	31	17	396	386	399	385	284	274	287	273	172	162	175	161
135	137	132	138	23	25	20	26	391	393	388	394	279	281	276	282	167	169	164	170
352	366	355	365	240	254	243	253	208	222	211	221	96	110	99	109	64	78	67	77
363	357	360	358	251	245	248	246	219	213	216	214	107	101	104	102	75	69	72	70
364	354	367	353	252	242	255	241	220	210	223	209	108	98	111	97	76	66	79	65
359	361	356	362	247	249	244	250	215	217	212	218	103	105	100	106	71	73	68	74
176	190	179	189	144	158	147	157	32	46	35	45	320	334	323	333	288	302	291	301
187	181	184	182	155	149	152	150	43	37	40	38	331	325	328	326	299	293	296	294
188	178	191	177	156	146	159	145	44	34	47	33	332	322	335	321	300	290	303	289
183	185	180	186	151	153	148	154	39	41	36	42	327	329	324	330	295	297	292	298

Figure 18. The $n_1 = 5$, $n_2 = 4$ composite square

$$
\begin{aligned}
&= n_1 n_2 (n_2^2 - 1)/2 + n_2^3 \times n_1 (n_1^2 - 1)/2 \\
&= n_1 n_2 (n_2^2 - 1 + n_2^2 n_2^2 - n_1^2)/2 \qquad . \\
&= n_1 n_2 (n_1^2 n_2^2 - 1)/2
\end{aligned}
$$

This holds for all rows in the composite quilt.

A similar argument holds for the columns and for the two principal diagonals.

To check for the broken diagonals, refer again to Figures 13 and 14. The required additions (which are all multiples of n_2^2) depend on the lengths of the segments—namely $n_2 - c$ and c, into which the broken diagonals starting at columns c in the top row are split.

When c is a multiple of n_2, the broken diagonals traverse just n_1 patches, each from its top left corner diagonally downward to its bottom right corner, and the sum of the numbers in these diagonals is the same as that in the principal diagonals, namely $n_1 m_2$.

When c is not a multiple of n_2, the additions (all of which are multiples of n_2^2) depend on the lengths of the segments, namely $n_2 - c$ and c, into which the broken diagonals starting at position c in the top row are split. To each of the n_1 segments of length $n_2 - c$ has been added a multiple of n_2^2, thus making a total addition to these segments of $(n_2 - c)n_2^2 m_1$. To each of the n_1 segments of length c has also been added a multiple of n_2^2, thus making a total addition to these segments of $cn_2^2 m_1$. Together, through the whole of the broken diagonal, the additions have therefore been

$$(n_2 - c + c)n_2^2 m_1 = n_2^3 m_1 .$$

This establishes that the sum of the numbers in any broken diagonal in the composite quilt is $n_1 m_2 + n_2^3 m_1$, which reduces, as shown earlier, to $n_1 n_2 (n_1^2 n_2^2 - 1)/2$, which is the magic constant m of the pandiagonal magic square of order $n_1 n_2$.

This proves that the numbers in the broken diagonals, as well as those in the rows, columns and principal diagonals of the new composite square, add to the magic constant and establishes the validity of the construction described.

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Constructing pandiagonal magic squares of arbitrarily large size

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In Part 1 of this feature article (*Mathematics Today* Vol:42 No1 February 2005,pp.23) a method is described for the construction of pandiagonal magic squares of order n for all prime values of $n > 3$. Here a slight change, together with an easy additional procedure when n is a multiple of 3, leads to a method of construction for **all odd** values of $n > 3$, not merely for primes. Enumeration of the squares that can be produced by this method (or other similar methods) is not discussed.

Paths

The method is based (as is that described in Part 1) on *paths* made by like numbers through the square, the paths being a series of *steps.* These steps are knight's moves in the game of chess. As described in Part 1, a path *wraps round* when it hits an edge of the square, continuing at the opposite edge. A path always returns to its starting position after n steps. There are steps other than knight's moves that can be used to form a path, some of which lead to the same squares, but these are not discussed here.

This method of paths is not new, but it is not well known. The patterns are evident in the 7 × 7 square discovered by Leonhard Euler (1707–83) – reproduced in Figure 1 – and the ground is well covered by Benson and Jacobi (1976) referred to in Part 1. Although not immediately obvious, this Euler square can itself be produced by using two compatible knight's paths and it has other properties in common with the squares discussed here.

(a)

40	7	16	3	22	32	48
25	34	47	35	9	17	1
10	15	4	27	33	42	37
28	44	38	8	18	6	26
20	5	21	30	45	36	11
43	39	13	19	0	23	31
2	24	29	46	41	12	14

(b)

55	10	22	03	31	44	66
34	46	65	50	12	23	01
13	21	04	36	45	60	52
40	62	53	11	24	06	35
26	05	30	42	63	51	14
61	54	16	25	00	32	43
02	33	41	64	56	15	20

Figure 1. Euler's pandiagonal magic square of order 7: (a) in decimal, (b) in base 7

Definitions

This Part comes as an extension of Part 1, where all relevant definitions are given: *normal pandiagonal magic squares*, *broken diagonals, the magic constant*, *auxiliary radix* and *unit orthogonal squares*, *paths* and *wrapping round.* The n^2 numbers that form the normal magic square consist of all the ordered pairs of numbers (written in base n), arranged so that each of the n numbers from 0 to n-1 appears on the left in tandem with each of the numbers from 0 to n-1 on the right, and vice versa.

Essential to the method is that the work is done using the numbers from 0 to n^2-1 (rather than from 1 to n^2) and in base n. By definition, each number appears just once in the completed square. This, in base n, means that each number has two components: the number (or symbol) on the left known as the *radix* component and the number (or symbol) on the right known as the *unit* component. The completed square can thus be split into two *auxiliary* squares consisting respectively of the numbers from 0 to $n-1$, each occurring n times, on the left of the number-pairs that form the radix auxiliary square; and the numbers from 0 to $n-1$, each likewise occurring n times, on the right of the number-pairs that form the unit auxiliary square.

If, when the two auxiliary squares combine, they form a square with each ordered number-pair occurring just once, they are said to be *orthogonal* and, when both auxiliary squares are pandiagonal magic, then the completed square is also pandiagonal magic.

The Construction

The method used (both here and in Part 1) to construct pandiagonal magic squares when n (> 3) is odd employs a pair of different knight's paths for the numbers that form the radix (left-hand) component and the numbers that form the unit (right-hand) component.

The paths chosen ensure that the repeated numbers from 0 to $n-1$ form two orthogonal auxiliary squares. The pair of knight's paths used in Part 1 for the radix and the unit auxiliary squares are, respectively, 'two rows down and one column across to the right' denoted by [2,1], and 'one row down and two columns across to the right' denoted by [1,2]. When n is a multiple of 3, these paths 'clash', the two auxiliaries are not orthogonal, and the process using these paths fails. The two paths $[r,c]$ and $[r',c']$ clash if there is a factor common to n and rc' – r'c (Brée, personal communication). For the paths [2,1] and [1,2], $rc' - r'c = 2\times 2 - 1\times 1 = 3$, which is a factor of $n = 9$. Here, the second path used in Part 1, namely [1,2], is replaced by the knight's path 'two rows up and one column across to the right', which we denote by [–2,1]. These two paths [2,1] and [–2,1] never clash when n is odd, as then $rc' - r'c = 2\times 1 + 1\times 2 = 4$, which cannot have a factor in common with an odd value of n. Thus, when n is odd, these paths lead to orthogonal auxiliary squares as demonstrated in Figures 2 and 3 where n = 5, 7, 9 and 15. The squares can thus be 'split' into two auxiliaries, whose construction and properties can be considered separately.

(a)

00	33	11	44	22
14	42	20	03	31
23	01	34	12	40
32	10	43	21	04
41	24	02	30	13

(b)

00	44	11	55	22	66	33
52	26	63	30	04	41	15
34	01	45	12	56	23	60
16	53	20	64	31	05	42
61	35	02	46	13	50	24
43	10	54	21	65	32	06
25	62	36	03	40	14	51

Figure 2. (a) A 5 x 5 pandiagonal magic square in base 5 and (b) a 7 x 7 pandiagonal magic square in base 7 constructed by using the knight's paths [2,1], [–2,1]

The radix and unit auxiliaries of this 7×7 square are

Radix

0	4	1	5	2	6	3
5	2	6	3	0	4	1
3	0	4	1	5	2	6
1	5	2	6	3	0	4
6	3	0	4	1	5	2
4	1	5	2	6	3	0
2	6	3	0	4	1	5

Unit

0	4	1	5	2	6	3
2	6	3	0	4	1	5
4	1	5	2	6	3	0
6	3	0	4	1	5	2
1	5	2	6	3	0	4
3	0	4	1	5	2	6
5	2	6	3	0	4	1

Figure 3. The radix and unit auxiliaries of the 7 x 7 square shown above in Figure 2

To construct the radix auxiliary when $n = 7$ in Figure 3 imagine an empty 7×7 square grid. Start (arbitrarily) with 0 in the top-left corner and place the other six 0s along the prescribed knight's path, here [2,1], wrapping round when the path hits the bottom row to the cell in the next-to-top row in the neighbouring column on the right. The seventh 0 occupies a cell in the right-most column and the next step (with a 'double' wrap round) would take the path back to its starting point in the top-left corner, making a complete circuit of the square. The other numbers from 1 to $n-1$ now have to be placed, in turn, along similar paths to fill the empty cells. Choose the starting position of 1 in the cell defined by the first step in the path [–2,l], and then place the remaining 1s along the path [2,1] mimicking that of the 0s (see Figure 3). Do the same with the 2s. Place the first 2 in the position 'two rows up and one row across to the right' from the first-placed 1, and place the remaining 2s along the path [2,1]. Follow the same procedure with the 3s, 4s, 5s and 6s to complete the radix square shown in Figure 3. The order in which numbers from 1 to 6 are placed along the path [–2,l] is arbitrary: their natural ascending order has been chosen here for clarity.

To construct the unit square follow the same procedure in reverse. With 0 in the top-left corner, place the other 0s in the path [–2,l]. Then place the numbers from 1 to 6 along the path [2,1] that starts with 0 in the top-left corner. Place the remaining 1s, 2s, 3s, 4s, 5s and 6s along paths [2,1].

Sequences

In the two orthogonal auxiliary squares formed by the knight's paths [2,1] and [–2,l] (see Figures 2 to 6) as also in Euler's square (Figure 1) which is formed by the knight's paths [2,–1] and [1,2], the numbers in the rows, in the columns and in the two opposing sets of parallel diagonals have related sequences determined by the positions of the 0s. They have identical top rows, namely

$$0 \quad (n+1)\,2 \quad 1 \quad 1+(n+1)/2 \quad 2 \quad 2+(n+1)/2 \quad \ldots$$
$$\ldots \quad r \quad r+(n+1)/2 \quad \ldots \quad (n-1) \quad (n-1)/2.$$

All the rows in both the radix and the unit squares follow the same sequence, starting at the 0 in each row and, reading from left to right, wrapping round when the right-hand edge is hit. This determines the positions of the numbers throughout both squares. The numbers in the columns of the square follow their own specific sequence, read either upward or downward. So, too, do the number in the diagonals as now explained.

The diagonals

In the 7×7 radix square in Figure 3, consider the diagonal containing the 0 that lies on the knight's path [2,1], two rows down and one column across from the top-left corner. This diagonal, read upward to the right, hits the top row in the fourth position from the top-left corner (and then wraps round to the bottom row). The next 0 along the path [2,1] lies in the parallel diagonal that hits the top row at the top-right corner, and so on for each parallel diagonal, wrapping round when hitting an edge. Thus successive parallel diagonals starting with 0 and read upward to the right hit the top row in positions that are 3 apart.

This is true for any squares of order n when n is a multiple of 3 and leads to a succession of numbers (in three circuits along the top row) that includes every number just once. Hence, each of the diagonals in both directions contains each number just once and the numbers add correctly to the magic constant.

The same argument applies to the unit auxiliary with directions reversed. This means that every diagonal (in both the radix and the unit squares) fulfils the conditions of being pandiagonal magic and so the completed square is also pandiagonal magic.

When n (>3) is a multiple of 3 the succession of every third number in the top row (read from left to right and wrapping round at the right-hand edge) leads to a set of three distinct sequences each occurring three times, namely

Table 1

0	3	6	9	12	.	.	.	$n-3$
1	4	7	10	13	.	. .	.	$n-2$
2	5	8	11	14	. .	.	.	$n-1$

(see Figures 4, 5, 7 and 8 where $n = 9$ and 15). These three sequences, together, contain each number from 0 to $n-1$ just once, but they do not have equal sums. This, however, can be corrected by one-to-one interchanges of numbers, as is now explained.

Interchanges when n = 9

When $n = 9$, the construction using the paths [2,1] and [–2,l] leads to the radix and unit auxiliary squares shown in Figure 4.

Radix

0	5	1	6	2	7	3	8	4
2	7	3	8	4	0	5	1	6
4	0	5	1	6	2	7	3	8
6	2	7	3	8	4	0	5	1
8	4	0	5	1	6	2	7	3
1	6	2	7	3	8	4	0	5
3	8	4	0	5	1	6	2	7
5	1	6	2	7	3	8	4	0
7	3	8	4	0	5	1	6	2

Unit

0	5	1	6	2	7	3	8	4
7	3	8	4	0	5	1	6	2
5	1	6	2	7	3	8	4	0
3	8	4	0	5	1	6	2	7
1	6	2	7	3	8	4	0	5
8	4	0	5	1	6	2	7	3
6	2	7	3	8	4	0	5	1
4	0	5	1	6	2	7	3	8
2	7	3	8	4	0	5	1	6

Figure 4. The orthogonal (from 0 to 8) 9 x 9 radix and unit auxiliary squares constructed by using the knight's paths [2,1] and [–2,1] respectively

Note that, as with $n = 5$ and $n = 7$, the radix and unit auxiliary squares are orthogonal and have identical top rows. Each number from 0 to 8 appears just once in each row, in each column and in the diagonals that run in one direction (from top-left downward to bottom-right in the radix square, and from top-right to bottom-left in the unit square). The diagonals that run in the opposite direction are of three kinds:

(i) the numbers 0 3 6 occurring three times,
(ii) the numbers 1 4 7 occurring three times, and
(iii) the numbers 2 5 8 occurring three times.

These particular sequences of three numbers, together containing each number from 0 to 8 just once are, as explained above, a consequence of choosing the numbers that followed the path [–2,1] in the first stage of the construction in their natural ascending order. There are just two ways in which the numbers from 0 to 8 can be arranged in three groups of three numbers with equal sums. Written in ascending order of magnitude they are

(a)	0	4	8	and	(b)	0	5	7
	1	5	6			1	3	8
	2	3	7			2	4	6

If, throughout both the original unadjusted auxiliary squares, the consecutive numbers from 0 to 8 are interchanged with the numbers written vertically below them in either (a) or (b) of the table below, this problem will be resolved.

Table 2

	0	1	2,	3	4	5,	6	7	8
(a)	0	1	2,	4	5	3,	8	6	7
(b)	0	1	2,	5	3	4,	7	8	6

Leave the numbers 0 1 2 unchanged and use Table 2 to effect interchanges between the other numbers. Then

0 3 6 become (a) 0 4 8 or (b) 0 5 7,
1 4 7 become (a) 1 5 6 or (b) 1 3 8,
and 2 5 8 become (a) 2 3 7 or (b) 2 4 6.

Either (a) or (b) changes 0 3 6, 1 4 7, 2 5 8 into three groups of three numbers, each group having the same sum, namely 12, which is one third of the magic constant as required. If the interchanges (a) are used then the radix and unit squares become as shown in Figure 5.

Radix

0	3	1	8	2	6	4	7	5
2	6	4	7	5	0	3	1	8
5	0	3	1	8	2	6	4	7
8	2	6	4	7	5	0	3	1
7	5	0	3	1	8	2	6	4
1	8	2	6	4	7	5	0	3
4	7	5	0	3	1	8	2	6
3	1	8	2	6	4	7	5	0
6	4	7	5	0	3	1	8	2

Unit

0	3	1	8	2	6	4	7	5
6	4	7	5	0	3	1	8	2
3	1	8	2	6	4	7	5	0
4	7	5	0	3	1	8	2	6
1	8	2	6	4	7	5	0	3
7	5	0	3	1	8	2	6	4
8	2	6	4	7	5	0	3	1
5	0	3	1	8	2	6	4	7
2	6	4	7	5	0	3	1	8

From	0	1	2	3	4	5	6	7	8
To	0	1	2	4	5	3	8	6	7

Figure 5. The 9 x 9 radix and unit squares after the interchanges

These corrected auxiliary squares can then be combined to form a pandiagonal magic square, which is shown in decimals in Figure 6.

This completes the discussion for $n = 9$.

0	30	10	80	20	60	40	70	50
24	58	43	68	45	3	28	17	74
48	1	35	11	78	22	61	41	63
76	25	59	36	66	46	8	29	15
64	53	2	33	13	79	23	54	39
16	77	18	57	37	71	47	6	31
44	65	51	4	34	14	72	21	55
32	9	75	19	62	38	69	49	7
56	42	67	52	5	27	12	73	26

Figure 6. The 9 x 9 pandiagonal magic square (from 0 to 80) in decimals

The interchanges required for odd multiples of 3 with n >9.

When n is a multiple of 3 the faulty diagonals that result when the paths [2,1] and [–2,1] are used to construct 'raw' radix and unit squares are shown in Table 1 above, each sequence of $n/3$ numbers being used three times to form a (faulty) diagonal. The numbers in all three sequences taken together run from 0 to $n-1$, each appearing just once. The numbers in the middle sequences add to the magic constant, but those in the top sequence add to a total that is $n/3$ too small, and those in the bottom sequence add to a sum that is $n/3$ too large. The special case when $n = 9$ has been dealt with above. It remains to find a way of adjusting the numbers in the top and the bottom sequences so that the numbers in all three add to the same total, the magic constant.

Note that, in Table 1, a number in the top sequence is 2 less than the number vertically below it in the bottom sequence. Interchanging them would add 2 to the sum of the numbers in the top sequence and subtract 2 from that in the bottom sequence. Likewise, a number in the top sequence is 5 less than the number in the bottom sequence one further position to the right. Interchanging them would add 5 to the sum of the numbers in the top sequence.

To make the sums of the numbers in all three sequences equal, the adjustment required is thus, first, to interchange a number (other than $n-3$, the last number in the top sequence) with a number one position to the right in the bottom sequence. This increases the sum of the numbers in the top sequence by 5 and decreases the sum of the numbers in the bottom sequence by 5. By definition $n/3$ is odd, so $(n/3-5)$ is even. It follows that further interchanges, now between a number or numbers in the top sequence with the number or numbers vertically below in the bottom sequence, will achieve the objective of making the sum of the numbers in all three sequences equal. When these interchanges are made to all the numbers throughout both the preliminary auxiliary squares, these squares still remain orthogonal and both will be pandiagonal magic. Linked together, they will thus form a pandiagonal magic square.

As illustration, use $n = 15$, the smallest odd number > 9 that is a multiple of 3. The radix and unit auxiliary squares constructed by using the knight's paths [2,1] and [–2,1] are as shown in Figure 7. The squares are divided into nine sub-squares to make reading the numbers (and tracing diagonals) easier. These sub-squares are labelled A, B, C and A', B', C' respectively merely to demonstrate the positions of the 5×5 sub-squares within the structure.

The sequences of numbers, each occurring three times in the faulty diagonals in the preliminary square, are given in Table 3.

Table 3

0	3	6	9	12
1	4	7	10	13
2	5	8	11	14

The numbers in the middle sequence add to 35, which is one-third of the magic constant. The numbers in the top and in the bottom sequences have a total that is $n/3 = 5$ too small and 5 too large respectively. Only one interchange is required. This can be 3 with 8, 6 with 11 or 9 with 14 . The result of the interchange of 9 and 14 is shown in Figures 7 and 8, where the adjusted 15×15 now pandiagonal magic auxiliary squares are shown. The final figure, Figure 9, depicts the 15×15 pandiagonal magic square constructed in the manner shown.

0	8	1	9	2	10	3	11	4	12	5	13	6	14	7
11	4	12	5	13	6	14	7	0	8	1	9	2	10	3
7	0	8	1	9	2	10	3	11	4	12	5	13	6	14
3	11	4	12	5	13	6	14	7	0	8	1	9	2	10
14	7	0	8	1	9	2	10	3	11	4	12	5	13	6
10	3	11	4	12	5	13	6	14	7	0	8	1	9	2
6	14	7	0	8	1	9	2	10	3	11	4	12	5	13
2	10	3	11	4	12	5	13	6	14	7	0	8	1	9
13	6	14	7	0	8	1	9	2	10	3	11	4	12	5
9	2	10	3	11	4	12	5	13	6	14	7	0	8	1
5	13	6	14	7	0	8	1	9	2	10	3	11	4	12
1	9	2	10	3	11	4	12	5	13	6	14	7	0	8
12	5	13	6	14	7	0	8	1	9	2	10	3	11	4
8	1	9	2	10	3	11	4	12	5	13	6	14	7	0
4	12	5	13	6	14	7	0	8	1	9	2	10	3	11

A	C	B
C	B	A
B	A	C

Figure 7. The radix auxiliary formed **by the path [2,1] when n=15**

This concludes the description of how an $n\times n$ pandiagonal magic square can be constructed when n (>3) is any odd number. There are many other ways of constructing these and other pandiagonal magic squares for different odd values of n, including 'composite squares' as described in Part 1 of this series of articles, but I will stop here, grateful for the help and patience of many friends.❑

0	8	1	9	2	10	3	11	4	12	5	13	6	14	7
4	12	5	13	6	14	7	0	8	1	9	2	10	3	11
8	1	9	2	10	3	11	4	12	5	13	6	14	7	0
12	5	13	6	14	7	0	8	1	9	2	10	3	11	4
1	9	2	10	3	11	4	12	5	13	6	14	7	0	8
5	13	6	14	7	0	8	1	9	2	10	3	11	4	12
9	2	10	3	11	4	12	5	13	6	14	7	0	8	1
13	6	14	7	0	8	1	9	2	10	3	11	4	12	5
2	10	3	11	4	12	5	13	6	14	7	0	8	1	9
6	14	7	0	8	1	9	2	10	3	11	4	12	5	13
10	3	11	4	12	5	13	6	14	7	0	8	1	9	2
14	7	0	8	1	9	2	10	3	11	4	12	5	13	6
3	11	4	12	5	13	6	14	7	0	8	1	9	2	10
7	0	8	1	9	2	10	3	11	4	12	5	13	6	14
11	4	12	5	13	6	14	7	0	8	1	9	2	10	3

A'	C'	B'
B'	A'	C'
C'	B'	A'

Figure 8. The unit auxiliary formed by the path [–2,1] when n=15

00	88	11	EE	22	AA	33	BB	44	CC	55	DD	66	99	77
B4	4C	C5	5D	D6	69	97	70	08	81	1E	E2	2A	A3	3B
78	01	8E	12	EA	23	AB	34	BC	45	CD	56	D9	67	90
3C	B5	4D	C6	59	D7	60	98	71	0E	82	1A	E3	2B	A4
91	7E	02	8A	13	EB	24	AC	35	BD	46	C9	57	D0	68
A5	3D	B6	49	C1	50	D8	61	9E	72	0A	83	1B	E4	2C
6E	92	7A	03	8B	14	EC	25	AD	36	B9	47	C0	58	D1
2D	A6	39	B7	40	C8	57	DE	62	9A	73	0B	84	1C	E5
D2	6A	93	7B	04	8C	15	ED	26	A9	37	B0	48	C1	5E
E6	29	A7	30	B8	41	CE	52	DA	63	9B	74	0C	85	1D
5A	D3	6B	94	7C	05	8D	16	E9	27	A0	38	B1	4E	C2
19	E7	20	A8	31	BE	42	CA	53	DB	64	9C	75	0D	86
C3	5B	D4	6C	95	7D	06	89	17	E0	28	A1	3E	B2	4A
87	10	E8	21	AE	32	BA	43	CB	54	DC	65	9D	76	09
4B	C4	5C	D5	6D	96	79	07	80	18	E1	2E	A2	3A	B3

Figure 9. An adjusted 15 x 15 pandiagonal magic square in base 15 (where A=10, B=11, C=12, D=13, E=14)