Editorial

A vacancy has arisen for membership of the *Mathematics Today* Editorial Board, as Hannah Davies recently resigned due to other commitments including doctoral research and a new job. We particularly invite expressions of interest from young, female and non-Caucasian members, in order to encourage gender equality and widening participation, though all self-nominations are welcome. Members of the Editorial Board are volunteers and help to guide the production of this publication. They meet twice per year in London and travel expenses are reimbursed. If you would like to apply for membership of the Board, please send a brief CV and covering letter to the Editorial Officer (`rebecca.waters@ima.org.uk`) by the end of April.

This January, I was browsing through a couple of excellent new books on modern applications [1] and popular science [2], while watching some rousing darts matches on television. One section related to the design of sporting tournaments which reminded me of a recent Urban Maths article [3] on tennis scoring and inspired me to investigate the formats of darts competitions.

The British Darts Organisation (BDO, founded 1973) and the Professional Darts Corporation (PDC, founded 1992) separately hold annual World Professional Darts Championships (WDC) around New Year. These are knockout tournaments for men, though the BDO holds similar events for women, and both finals formats comprise the best of 13 sets, each of which is the best of five legs. Thus, the first player to reach a score of 501 ending with a double or bullseye wins each leg, the first player to win three legs wins each set and the first player to win seven sets wins the match.

Table 1: Professional darts tournament formats

Organiser	Tournament	Sets	Legs
BDO	BDO World Darts Championship	13	5
	Winmau World Masters	15	3
	BDO World Trophy	1	25
PDC	William Hill WDC	13	5
	Coral UK Open	1	21
	BetVictor World Matchplay	1	35
	PartyPoker.com World Grand Prix*	9	5
	Singha Beer Grand Slam of Darts	1	31

*double or bullseye to start

Table 1 displays the finals formats for these and other major tournaments organised by the BDO and PDC, and my aim is to compare these schemes for fairness, balance and efficiency. Fairness ensures that the best players usually win, balance ensures exciting matches with uncertain outcomes, and efficiency ensures small mean and standard deviation of match durations.

Consider a match between players A and B, which comprises the best of s sets, each of which is the best of l legs. Generally, the rules allow for a tie-breaker in the final set if the scores are very close, though we ignore this inconvenience here. The first player to throw is decided as whoever lands a single dart closer to the bullseye, after which the first throw alternates among legs within sets and among sets within the match. Define events:

- L_1 = 'A wins leg in which he threw first';
- L_2 = 'A wins leg in which he threw second';

and similar events for sets (S_1 and S_2) and matches (M_1 and M_2). Reasonably assuming that the players' performances remain constant during play, we also define probabilities $\pi_1 = P(L_1)$ and $\pi_2 = P(L_2) \le \pi_1$. Figure 1 illustrates how to interpret combinations of π_1 and π_2. Our main challenge is to evaluate $P(S_1)$ and $P(S_2)$, which then enable us to evaluate $P(M_1)$ and $P(M_2)$.

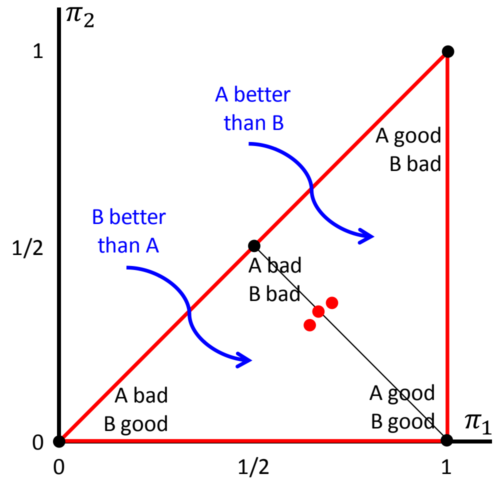

Figure 1: Interpretation of (π_1, π_2) combinations

To illustrate the analysis involved, consider a set that comprises the best of $l = 5$ legs. There are

$$\#(S) = \sum_{i=1}^{3} C(5-i, 2) = 10$$

ways in which Player A can win the set, where i represents the winning margin. These include one case where $i = 3$, three cases where $i = 2$ and six cases where $i = 1$. For example, the three cases where $i = 2$ progress thus: (a) 1–0, 2–0, 2–1, 3–1; (b) 1–0, 1–1, 2–1, 3–1; (c) 0–1, 1–1, 2–1, 3–1. With so few terms, we can evaluate the probabilities that Player A wins the set, after some algebraic manipulation:

$$P(S_1) = 6\pi_1^3\pi_2^2 - 6\pi_1^3\pi_2 - 9\pi_1^2\pi_2^2 + \pi_1^3 + 6\pi_1^2\pi_2 + 3\pi_1\pi_2^2;$$
$$P(S_2) = 6\pi_2^3\pi_1^2 - 6\pi_2^3\pi_1 - 9\pi_2^2\pi_1^2 + \pi_2^3 + 6\pi_2^2\pi_1 + 3\pi_2\pi_1^2.$$

Figure 2 displays contour plots of $P(S_1)$ and $P(S_2)$ respectively for different combinations of the probabilities π_1 and π_2. Note that the formula for $P(S_2)$ is the same as that for $P(S_1)$ if

Table 2: Simulation results based on 10^6 matches

Tournament	(s,l)	$\pi_1 = 2/3,\ \pi_2 = 1/3$ $P(M_1)$	Legs/Match	$\pi_1 = 0.70,\ \pi_2 = 0.35$ $P(M_1)$	Legs/Match	$\pi_1 = 0.65,\ \pi_2 = 0.30$ $P(M_1)$	Legs/Match
BDO World Darts Championship	(13,5)	0.51	47	0.66	46	0.37	46
Winmau World Masters	(15,3)	0.52	33	0.64	33	0.39	33
BDO World Trophy	(1,25)	0.53	22	0.63	22	0.42	22

Glen Durrant celebrates with the 2017 BDO World Championship trophy.
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we swap π_1 and π_2, which explains why the contours in the two graphs are complementary reflections about the line $\pi_1+\pi_2=1$.

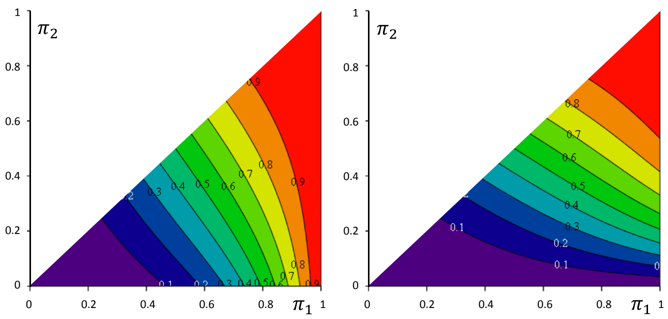

Figure 2: Probabilities that Player A wins a set of 5 legs

Now consider the number of ways in which Player A can win a match that comprises the best of s sets, each of which is the best of l legs. Generalising and extending the above formula for $\#(S)$ gives the result:

$$\#(M) = \sum_{i=1}^{(s+1)/2} C\left(s-i, \frac{s-1}{2}\right) \sum_{j=1}^{(l+1)/2} C\left(l-j, \frac{l-1}{2}\right).$$

For the three BDO tournaments in Table 1, $\#(M)$ takes these values: 17,160; 19,305; 5,200,300. Clearly, we cannot evaluate $P(M_1)$ and $P(M_2)$ analytically and must proceed by simulating matches computationally. In each case, I simulated 10^4 matches in Excel assuming that Player A throws first initially. However, a kind reviewer simulated 10^6 matches using Python for improved accuracy and reliability.

We consider three scenarios to illustrate typical results that this analysis provides, as identified by the red dots (Figure 1). The first case has Players A and B of equal ability with $\pi_1+\pi_2=1$, the second has A slightly better than B with $\pi_1+\pi_2=1.05$, and the third case has B slightly better than A with $\pi_1+\pi_2=0.95$. Table 2 presents the results of this preliminary analysis. Although these calculations assume steady performances, represent only three (π_1,π_2) combinations, ignore tie-breaks and consider only three tournament formats, they nevertheless generate some interesting observations.

Regarding the fairness criterion, the $(13,5)$ format is best in all three cases. This is because its value for $P(M_1)$ is nearest to one half when the players are equally skilled, greatest when A is better than B, and least when B is better than A. The balance criterion would ideally set $P(M_1)=1/2$ for all values of π_1 and π_2, and the $(1,25)$ format is best in this regard. However, this conflicts strongly with the fairness criterion, which would ideally set $P(M_1)=1$ whenever $\pi_1+\pi_2>1$ and $P(M_1)=0$ whenever $\pi_1+\pi_2<1$.

Clearly, a tournament that does not reward skill would soon fall out of favour among the better players, who would prefer the fairness of $(13,5)$ to the balance of $(1,25)$. Perhaps a compromise such as the $(15,3)$ format is needed, or a variety of tournaments as currently exists. The efficiency criterion considers the average numbers of legs per match as simulated and presented in Table 2. The $(1,25)$ format consistently involves the shortest matches and so is the most convenient for schedulers and broadcasters.

Enough of this indulgence and on to important matters! Those who attended the *IMA Mathematics 2017 Conference* in London on 23 March witnessed some excellent presentations in a convivial setting. This was the 11th in a series of annual IMA one-day meetings, which proved to be equally popular and enjoyable. Another exciting gathering now appears on the horizon: following the success of the *IMA Festival of Mathematics and its Applications* at the University of Manchester in 2014, the University of Greenwich will host a similar festival on 27–28 June this year (see page 47). Our Vice President (Communications), Noel-Ann Bradshaw, is organising this event and has attracted some of the UK's top maths communicators, so join them for lots of fun mathematics if you can.

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References

1 Haigh, J. (2016) *Mathematics in Everyday Life*, Springer, Switzerland.

2 Matthews, R. (2016) *Chancing It: the Laws of Chance and How They can Work for You*, Profile Books, United Kingdom.

3 Townie, A. (2015) Urban Maths: Point Scoring, *Mathematics Today*, vol. 51, no. 5, pp. 238–240.