Historical Notes: The Rise and Fall of the Consistency Programme

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On Wednesday, 8 August 1900, David Hilbert addressed a Paris audience who had gathered for the keynote speech of the *Second International Congress of Mathematicians* (ICM). Laying out what are now known as Hilbert's problems, the Paris address gives us an insight into what were seen as some of the outstanding mathematical problems facing the mathematical community at the turn of the 20th century [1].

The Second ICM was held in Paris alongside the Exposition Universelle, symbolised by this Ferris wheel.

However, the Paris address does not simply tell us what was important for the mathematicians present. As an address from a leading mathematician at what was arguably the world's leading mathematical centre, it can also be seen as a lecture concerning what they *should* regard as such. As much as it is a comment on the state of mathematics in 1900, Hilbert's Paris address serves as a programmatic document for how it should subsequently develop.

Several themes arise from the Paris address, which would play a central role in Hilbert's career for the next 30 years. Starting his address with a call for a new 'rigor' in mathematics – which he identified as nothing more than the 'requirement of logical deduction by means of a finite number of processes' for any given mathematical question – Hilbert went on to call explicitly for the axiomatisation of the various mathematical disciplines [1, p. 441].

Axiomatisation, however, along with its methodological counterpart in finite deductive reasoning, was not simply encouraged for its methodological superiority. It was not just the 'simpler and the more easily comprehended' [1, p. 441] method of mathematical proof, as Hilbert understood it as having important consequences for mathematical knowledge.

For Hilbert, the axiomatic method was linked to two claims. The first was the decidability of every well-phrased statement within an axiomatised mathematical theory. This he termed the 'axiom of the solvability', which stated that for any mathematical question its 'solution must follow by a finite number of purely logical processes' [1, p. 445]. The second was that in addition to being possible, the 'most important' result for any formal system was that the axioms be demonstrably [1, p. 447]:

> Not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results.

Axiomatic reasoning would, therefore, demonstrate that nothing was beyond the grasp of mathematicians and that the knowledge they acquired was safe. This was not an abstract claim. For Hilbert, consistency and solvability jointly served to defend the status of scientific knowledge in general and modern mathematical methods in particular.

In 1872, the physiologist Emil du Bois-Reymond had declared in an address to the German Society of Naturalists and Physicians that science was marked by *ignorabimus* – the modern scientist had to accept the limits of their method. Likewise, in mathematics there remained scepticism towards modern, set-theoretic methods and, in particular, the use of the transfinite. A consistency proof, in Hilbert's eyes, would render [1, p. 448]:

> The doubts which have been expressed occasionally as to the existence of the complete system of real numbers … totally groundless.

It would thereby put analysis on a stable footing. The axiomatic method and the axiom of solvability would together demonstrate that 'in mathematics there is no *ignorabimus*' [1, p. 445].

Paradoxes and the defence of set theory

Hilbert's Paris address laid out themes that would serve as constants in debates on formal mathematics. These were made more pressing by the publication of Russell's paradox in 1902, which demonstrated that set theory, in its *naive* form, was inconsistent. While Hilbert was both aware of the paradoxes and their severity prior to 1900 [2], by 1904 the need for a consistency proof for set theory and arithmetic had become more pressing. Although Hilbert had claimed in 1900 that a consistency proof would demonstrate the 'existence' of the real numbers [1, p. 448], by 1904 it was known that concepts regarded as unproblematic in Cantorian set theory had rendered the theory contradictory. The goal of axiomatisation had become one of not just demonstrating the 'mathematical existence' of the transfinite but the viability of modern set-theoretic methods themselves [2, p. 103].

For Hilbert and those aligned with him, this meant that a consistency proof was imperative. Maintaining the positive role that axiomatic reasoning had to play in mathematics, he argued that 'all the difficulties' that had affected the theories of Cantor, Dedekind and Frege – the latter of these had unsuccessfully attempted to reduce arithmetic to logic – could 'be overcome' through the axiomatic method [3, p. 131]. An axiomatisation that avoided contradiction was a necessary but insufficient step in demonstrating set theory's viability.

Motivated by the contradictions that arose when axiomatising set theory, Zermelo acknowledged in 1908 that while his axiomatisation did not lead to any paradoxes, it was 'essential' (*wesentlich*) that a consistency proof be given for it [4, p. 262]. This claim was echoed by Hilbert 10 years later: 'to restore the reputation of mathematics' it was 'not enough merely to avoid the existing contradictions', it had to be shown that 'contradictions based on the underlying axiom-system are *absolutely impossible*' [5, p. 1112].

The need to defend what has become known as *classical mathematics* was accentuated by the rise of an oppositional movement to set theory and analysis within the mathematical community. While criticism of set-theoretic methodology was as old as the theory, debates in the late 19th century and early 20th century had been relatively calm, as indicated by Hilbert's reference to the occasional expression of doubt regarding the definition of the real numbers.

In the 1920s, the debates became increasingly severe. With the publication of Hermann Weyl's 'On the new foundational crisis of mathematics' in 1921 [6], the intuitionist movement led by L.E.J. Brouwer began to attack Hilbert's programme on its own terms. Attacking the axiom of solvability as 'Hilbert's dogma' [7, p. 102], they pressed formalism on its failure to deliver a consistency proof, and questioned how compelling one would be, even if given. In 1927, Brouwer was to observe, correctly, that in the absence of a consistency proof, when it came to the foundations of mathematics, on formalism's own terms, 'nothing [had] been secured' [7, p. 541].

Incompleteness and the end of formalism

The intuitionistic critique challenged formal mathematics on its weakest point: no consistency proof existed, despite set theory having been axiomatised almost 20 years earlier. In 1927 and 1928, however, the debate looked very different, and by 1928, Hilbert likely believed that formalism was on course for total victory. With the debate on foundations having fractured politically, he had succeeded in removing his intuitionist opponents from the board of Germany's leading journal, *Mathematische Annalen*, due to accusations of nationalist 'blackmail', after they attempted to organise a boycott of the Bologna ICM [7, pp. 551]. This ended Brouwer's public career and left the intuitionist movement largely non-combative.

At the same time, Hilbert claimed that a consistency proof for arithmetic was essentially complete and that 'only the task of carrying out a purely mathematical proof of finiteness' remained [8, p. 479] (a claim he would repeat in [9, p. 1154]). With his opponents marginalised and a consistency proof as good as given, formalist victory seemed assured.

Speaking in September 1930 in Königsburg to the same society to which du Bois-Reymond had declared the *ignorabimus* in 1872, Hilbert railed against [9, p. 1165]:

> Those who today, with a philosophical air and a superior tone, prophesy the downfall of culture and fall into an *ignorabimus* … our answer is on the contrary: We must know. We shall know.

It was not to be. Unknown to Hilbert, one day prior and in the same city, Kurt Gödel had announced a proof stating that in any consistent system strong enough to produce Peano arithmetic, there were statements that were true but unprovable within it. The incompleteness theorems were published two months later. The first demonstrated that the axiom of solvability was false: within a formal system, there was an *ignorabimus* after all. The second demonstrated that such a system could not prove its own consistency. Two of the central tenets of Hilbert's programme since 1900, one of which, consistency, had been regularly declared a prerequisite to the establishment set-theoretic mathematics as legitimate, had been proven impossible.

The axiomatic method could not guarantee what Hilbert and his acolytes said it could, and his programme had failed. Yet neither set theory nor the fields it is seen to serve as a foundation for were given up. Set theory is still taught as the theory that grounds the rest of the mathematical framework, and introductory books present Zermelo–Fraenkel set theory with choice (ZFC) as having saved the theory from contradiction.

Yet, this is not how it was seen at the time: the goal of axiomatisation was to demonstrate that no further contradictions could be found, not to demonstrate that the ones we already know of are blocked. The incompleteness theorems ensured that this crucial goal could not be reached. Yet, with opposition to set theory marginalised, rather than marking a new beginning, Gödel's incompleteness theorems marked the end of a period of mathematical crisis. The mathematics that formalists sought to defend survived the internal failure of the project that they had used to justify it.

This episode stands in tension with how we might expect a mathematical dispute to be conducted and resolved. Despite having failed to meet the criteria it set for its own success, set theory retained, and still retains, its foundational role. Reflecting on the end of the formalist project and intuitionism's marginalisation before it, Dirk van Dalen writes that 'although Hilbert had won the conflict in the social sense', on account of incompleteness 'he had lost it in the scientific sense' [7, p. 591].

However, one might wonder whether this distinction is tenable: how coherent is a distinction between the scientific and the social if it was set theory's social victory that helped it weather what its advocates and critics alike would have regarded as a scientific defeat?

References

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